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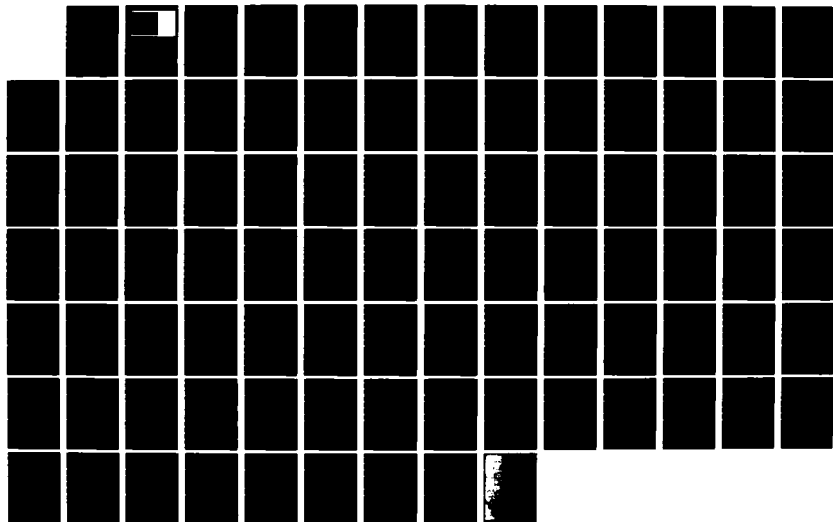
AN INTERFACE TRACKING ALGORITHM FOR THE POROUS MEDIUM  
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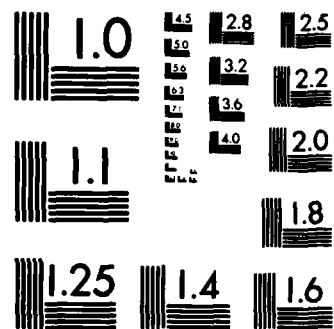
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AN INTERFACE TRACKING ALGORITHM  
FOR THE POROUS MEDIUM EQUATION

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E. DiBenedetto\* and D. C. Hoff\*

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ABSTRACT

*For an initial value problem*  
We study the convergence of a finite difference scheme for the Cauchy problem for the porous medium equation,  $u_t = (u^m)_{xx}$ ,  $m > 1$ .

The scheme exhibits the following two features. The first is that it employs a discretization of the known interface condition for the propagation of the support of the solution. *This generates* ~~We~~ thus generate approximate interfaces as well as an approximate solution.

The second feature is that it contains a vanishing viscosity term. This term permits an estimate of the form  $\|(u^{m-1})_{xx}\|_{1,R} < c/t$ .

*This is* We prove that both the approximate solution and the approximate interfaces converge to the correct ones.

Finally error bounds for both solution and free boundaries are proved in terms of the mesh parameters.

AMS (MOS) Subject Classification: 35K65, 35Q20, 35R35, 39A12, 39-04.

Key Words: Porous medium equation, finite speed of propagation, free boundary numerical solution, finite difference methods, algorithm, and error bounds.

Work Unit Number 3 - Numerical Analysis

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## SIGNIFICANCE AND EXPLANATION

The porous medium equation represents a model for the flow of a gas in a medium with constant porosity. The most interesting feature of such a physical phenomenon is that if initially the gas is concentrated in a compact set, then for later times even though the support expands, it remains compact. The boundary of the region where the gas is concentrated is the free boundary of the problem.

In this paper we construct numerically the solution of such a problem and also the free boundary.

We also supply error bounds in terms of the mesh parameters, both for the solution and the interfaces in the sup norm.

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# AN INTERFACE TRACKING ALGORITHM FOR THE POROUS MEDIUM EQUATION

E. DiBenedetto\* and D. C. Hoff\*

## 1. Introduction

In this paper we derive and analyze a finite difference scheme for computing both the solution and the interfaces for the porous medium equation in one space dimension. We demonstrate that the approximate solutions and the approximate interface curves converge to the correct ones, and obtain  $L^\infty$  bounds for the error in terms of the mesh parameter.

Consider the laminar flow of a polytropic fluid of density  $(x,t) \rightarrow u(x,t)$ , in a porous medium which is assumed to occupy the whole space, and suppose that at time  $t = 0$  the fluid is contained in the slab  $\zeta_\ell(0) \leq x \leq \zeta_r(0)$ . The phenomenon can be modeled by

$$(1.1) \quad u_t = (u^m)_{xx}, \quad (x,t) \in S_T \equiv \mathbb{R} \times (0,T] \\ 0 < T < \infty,$$

$$(1.2) \quad u(\cdot, 0) = u_0(\cdot) \quad \text{in } \mathbb{R},$$

where  $m > 1$  is a given constant, and  $u_0$  is a given non-negative function such that  $u_0(x) > 0$  if  $x \in (\zeta_\ell(0), \zeta_r(0))$  and  $u_0(x) = 0$  elsewhere. We assume  $u_0$  is continuous in  $\mathbb{R}$ .

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Since the problem is degenerate, (1.1)-(1.2) is interpreted in a weak sense and the solution possesses a modest degree of regularity. Precisely  $(x,t) \rightarrow u(x,t)$  is said to be a weak solution of (1.1)-(1.2) if

$$u \in C(S_T) \quad ; \quad (u^m)_x \in L^2(S_T)$$

and

$$(1.3) \quad \int_{\mathbb{R}} u(x, \cdot) \phi(x, \cdot) dx \Big|_{t_0}^t + \\ + \int_{t_0}^t \int_{\mathbb{R}} [-u \phi_t + (u^m)_x \phi_x] dx dt = 0$$

for all  $\phi$  satisfying

$$(1.4) \quad \begin{cases} \phi \in H^1(S_T) \cap L^\infty(S_T) & \text{and} \\ x \rightarrow \phi(x, t) & \text{is compactly supported in } \mathbb{R} \\ \text{uniformly in } t & , \end{cases}$$

and for all intervals  $[t_0, t] \subseteq [0, T]$ .

The pressure  $v$  in the fluid is connected to the density by

$$(1.5) \quad v = u^{m-1}$$

up to some multiplicative constant, and it satisfies

$$(1.6) \quad v_t = m v v_{xx} + \frac{m}{m-1} (v_x)^2, \quad \text{in } S_T$$

$$(1.7) \quad v_0(\cdot) = v(\cdot, 0) = u_0^{m-1}.$$

The Cauchy problem (1.6)-(1.7) is also interpreted in the weak sense

$$v \in C(S_T) \quad ; \quad v_x \in L^2(S_T)$$

and

$$(1.8) \quad \int_{\mathbb{R}} v(x, \cdot) \phi(x, \cdot) \Big|_{t_0}^t dx + \\ + \int_{t_0}^t \int_{\mathbb{R}} [-v \phi_t + m v v_x \phi_x + \frac{m(m-2)}{m-1} (v_x)^2 \phi] dx dt = 0$$

for all  $\phi$  satisfying (1.4), and all intervals  $[t_0, t] \subset [0, T]$ .

Existence and uniqueness of weak solutions of (1.1)-(1.2) was first proved by Oleinik-Kalashnikov and Chouli-Lin in [15], and the equivalence of (1.3) and (1.8) is due to Aronson [2].

A consequence of the degeneracy is that  $u(\cdot, t)$  and  $v(\cdot, t)$  are supported in a finite interval  $[\zeta_\ell(t), \zeta_r(t)]$ .

The curves  $(t, \zeta_\ell(t))$ ,  $(t, \zeta_r(t))$ , which we refer to as the left and right interfaces are Lipschitz continuous and monotone decreasing and increasing respectively (see [3]). The interface curves and the pressure  $v$  are connected by the Stefan-like conditions (see [3, 11])



$$(1.9) \quad \begin{cases} \lim_{t \nearrow \zeta_r} v_x(x, t) = -\frac{m-1}{m} \zeta'_r(t) \\ \lim_{t \searrow \zeta_\ell(t)} v_x(x, t) = -\frac{m-1}{m} \zeta'_\ell(t) \end{cases} .$$

It should be noted that conditions (1.9) are not part of the original problem, but rather are known to be satisfied by the unique solution of (1.1)-(1.2).

Nevertheless our algorithm will be based upon suitable discretization of both (1.6) and (1.9).

We now give a detailed description of our algorithm.

Let  $\Delta x$  and  $\Delta t$  denote increments in  $x$  and  $t$ , and let

$$x_k = k\Delta x, \quad k \in \mathbb{Z}; \quad t_n = n\Delta t, \quad n \in \mathbb{N} \cup \{0\}.$$

The approximations to  $v(x_k, t_n)$ ,  $\zeta_\ell(t_n)$  and  $\zeta_r(t_n)$  will be denoted by  $v_k^n$ ,  $\zeta_\ell^n$ ,  $\zeta_r^n$  respectively.

Actually we shall describe the computations only for the right hand interface; the computations near  $\zeta_\ell^n$  are completely analogous. We therefore suppress the subscript and denote  $\zeta_r^n$  by  $\zeta^n$ .

To start the scheme let  $v_k^0 = v_0(x_k)$  and  $\zeta^0 = \zeta(0)$ . Next define

$$K(0) = \max \{k : x_{k+1} \leq \zeta^0\}$$

and

$$s_0 = \zeta^0 - x_{K(0)} .$$

Then in analogy with (1.9) we compute  $\zeta^1$  from the equation

$$\zeta^1 = \zeta^0 + \frac{m}{m-1} \frac{v_{K(0)}^0}{s_0} \Delta t .$$

Observe that  $s_0 \geq \Delta x$  and  $\zeta^1 \geq \zeta^0$ .

Now given  $\zeta^{n+1} \geq \zeta^n$  and  $v_j^n$  for  $j \in \mathbb{Z}$ , we proceed as follows. First define

$$(1.10) \quad \begin{cases} K(n+1) = \max \{k : x_{k+1} \leq \zeta_r^n\} \\ \bar{K}(n+1) = \min \{k : x_{k-1} \geq \zeta_\ell^n\} \end{cases} ,$$

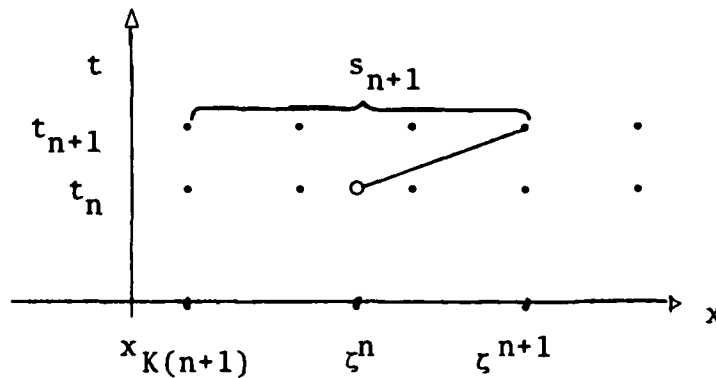


Fig. 1.

Then for  $\bar{K}(n+1) \leq k \leq K(n+1)$  compute  $v_k^{n+1}$  from the finite difference equation

$$(1.11) \quad \frac{v_k^{n+1} - v_k^n}{\Delta t} = m(v_k^n + \epsilon) \frac{v_{k+1}^n - 2v_k^n + v_{k-1}^n}{(\Delta x)^2} + \frac{m}{m-1} \left( \frac{v_{k+1}^n - v_{k-1}^n}{2\Delta x} \right)^2 ,$$

where  $\epsilon > 0$  will be chosen later. Observe that we do not enforce the difference equation across the interface.

Next let

$$(1.12) \quad s_{n+1} = \zeta^{n+1} - x_{K(n+1)} ,$$

and observe that

$$(1.13) \quad \Delta x \leq s_{n+1} .$$

Then for  $x_{K(n+1)} \leq x_k \leq \zeta^{n+1}$ , compute  $v_k^{n+1}$  from the linear interpolation

$$(1.14) \quad v_k^{n+1} = \frac{\zeta^{n+1} - x_k}{s_{n+1}} v_{K(n+1)}^{n+1} .$$

Finally set  $v_k^n = 0$  for  $x_k \geq \zeta^{n+1}$  and compute  $\zeta^{n+2}$  from

$$(1.15) \quad \zeta^{n+2} = \zeta^{n+1} + \frac{m}{m-1} \frac{v_{K(n+1)}^{n+1}}{s_{n+1}} \Delta t .$$

We shall prove that  $v_k^n \geq 0$  for all  $n$  and  $k$  so that by (1.15)  $\zeta^{n+2} \geq \zeta^{n+1}$ . Thus the support of the approximate solution increases monotonically in  $t$ .

In addition, the fact that  $s_n \geq \Delta x$  insures that numerical instabilities are avoided in the computations (1.14) and (1.15).

Introducing the notations

$$\beta = \frac{\Delta t}{(\Delta x)^2} \quad ; \quad Av_k = v_{k+1} - 2v_k + v_{k-1} \quad ,$$

we can rewrite the difference scheme (1.11) in the form

$$(1.16) \quad v_k^{n+1} = v_k^n + m\beta(v_k^n + \epsilon)Av_k^n + \frac{m\beta}{m-1} \left( \frac{v_{k+1}^n - v_{k-1}^n}{2} \right)^2 .$$

We shall assume throughout that

$$[A1] \quad 0 \leq v_0(x) \leq M \quad , \quad \forall x \in \mathbb{R} \quad ;$$

$$[A2] \quad |v_0(x) - v_0(y)| \leq \gamma_0 |x - y| \quad \forall x, y \in \mathbb{R} \quad ;$$

$$[A3] \quad \epsilon \text{ is of the order of } \Delta x \text{ and}$$

$$\epsilon \geq 9 \frac{m+2}{m-1} \gamma_0 \Delta x \quad ;$$

$$[A4] \quad 2m\beta [M + \epsilon + \frac{m}{m-1} \gamma_0 \Delta x] \leq 1 \quad ,$$

where  $M$  and  $\gamma_0$  are given positive constants. Since  $\epsilon = O(\Delta x)$ , condition [A4] on  $\beta$  is seen to be a slight strengthening of the usual parabolic stability condition.

We let  $h$  denote the pair  $(\Delta x, \Delta t)$ , and we construct approximate solutions  $v^h$  and approximate interface curves  $\zeta_\ell^h$  and  $\zeta_r^h$  by piecewise linear interpolation. Our results

may be summarized as follows

$$[I] \quad \|v^h - v\|_{\infty, S_T} \leq C(T) (\Delta x)^p ,$$

$$[II] \quad v_x^h \longrightarrow v_x \quad \text{in } L^q(S_T) , \quad \text{for all } q \in [1, \infty)$$

$$[III] \quad \|(\zeta_\ell^h, \zeta_r^h) - (\zeta_\ell, \zeta_r)\|_{\infty, [0, T]} \leq C(T) (\Delta x)^{p/2} ,$$

where  $p$  is defined in terms of  $m$  in Theorem 4.1 below. Further comments will be made in Section 6 about these rates of convergence, where we present and discuss the results of some numerical experiments.

The idea of exploiting an interface condition such as (1.9) for computational purposes seems to have been first used by Hüber [10] in connection with the one-phase Stefan problem (see also [7]).

We remark on the introduction of the vanishing viscosity  $\epsilon$ . If  $\epsilon$  were zero, the continuous analog of (1.11)-(1.15) would be overspecified. The artificial viscosity  $\epsilon$  thus seems to stabilize our finite difference scheme. More specifically, the presence of the  $\epsilon$  allows us to derive a lower bound for  $v_{xx}^h$  (in the sense of distributions). This in turn yields a uniform modulus of semicontinuity for  $v_x^h$  and via the interface condition, for  $\zeta_t^h$ . It is this semicontinuity which is crucial in proving the convergence of the approximate interface curves, as well as in estimating

the rate of convergence.

We briefly comment on related, at our knowledge available results. In [9] Gravelleau and Jamet obtained solutions of the porous medium equation and related equations by employing a difference scheme similar to ours. However their scheme is applied in all of  $\{t \geq 0\}$  so that approximate interfaces are not computed. Moreover numerical evidence indicates that the supports of their approximate solutions spread out too rapidly in time. Thus computing the interfaces by "shock capturing" seems to be unsatisfactory.

While this paper was in preparation, Tomoeda and Mimura [13] informed us that they have recently derived an interface tracking algorithm for the porous medium equation. Numerical evidence suggests that the approximate interfaces computed by their scheme are accurate, but they are unable to prove this result.

In addition their scheme is somewhat complicated to implement, since it involves solving Riemann problems for the Burgers equation at each mesh point. Both their scheme and ours suffer from the parabolic stability condition  $\Delta t = O[(\Delta x)^2]$ .

The paper is organized as follows. Section 2 contains the derivation of basic estimates. Specifically we prove the finite difference analog of the following facts, which are known to hold for the exact solution  $v$  of (1.6)-(1.7):

$$(1.17) \quad 0 \leq v \leq M$$

$$(1.18) \quad \|v_x\|_{\infty, S_T} \leq \gamma_0 \quad (\text{see [2]})$$

$$(1.19) \quad |v(x, t_2) - v(x, t_1)| \leq C |t_2 - t_1|^{1/2} \quad (\text{see [8,12]})$$

$$(1.20) \quad \begin{cases} v_{xx} \geq -\frac{m-1}{m(m+1)} \frac{1}{t} & \text{and} \\ \|v_{xx}(\cdot, t), v_t(\cdot, t)\|_{1, R} \leq \frac{C}{t} & (\text{see [5]}) \end{cases} .$$

In Section 3 we demonstrate the convergence of the approximate solutions and interfaces to the correct ones by making use of various compactness arguments.

The error estimates are proved in Sections 4 and 5.

Finally in Section 6 we present and discuss the results of several numerical experiments.

Throughout the paper we make the convention that  $C$  shall denote a generic positive constant depending only on  $m$ ,  $M$ ,  $\gamma_0$  and some specified time  $T$ .

## 2. Basic Estimates

We begin the analysis by establishing maximum principles for  $v_k^n$  and for the discrete space derivative

$$w_k^n = \frac{v_k^n - v_{k-1}^n}{\Delta x} .$$

We assume throughout that the initial function  $v_0$  satisfies assumptions [A1] and [A2], and that the mesh parameters  $\varepsilon$  and  $\beta$  satisfy [A3] and [A4].

Lemma 2.1    The bounds

$$(2.1) \quad 0 \leq v_k^n \leq m$$

and

$$(2.2) \quad |w_k^n| \leq \gamma_0$$

hold for all  $k$  and all  $n \geq 0$  .

**Proof.** The results hold for  $n = 0$  by hypothesis. Proceeding by induction on  $n$  , we rewrite the last term in the difference equation (1.16) as



$$\frac{m\Delta t}{m-1} \left( \frac{w_{k+1}^n + w_{k-1}^n}{2} \right) \left( \frac{v_{k+1}^n - v_{k-1}^n}{2\Delta x} \right) .$$

Rearranging, (1.16) thus becomes

$$\begin{aligned} (2.3) \quad v_k^{n+1} &= [1 - 2m\beta(v_k^n + \epsilon)]v_k^n \\ &+ [m\beta(v_k^n + \epsilon) + \frac{m}{m-1} \frac{\beta\Delta x}{4} (w_{k+1}^n + w_k^n)]v_{k+1}^n \\ &+ [m\beta(v_k^n + \epsilon) - \frac{m}{m-1} \frac{\beta\Delta x}{4} (w_{k+1}^n + w_k^n)]v_{k-1}^n . \end{aligned}$$

Using the induction hypotheses (2.1) and (2.2), we have that the coefficients of  $v_k^n$  and  $v_{k+1}^n$  in this expression are bounded below by

$$1 - 2m\beta(M + \epsilon) \quad \text{and} \quad m\beta(\epsilon - \frac{\gamma_0 \Delta x}{2(m-1)})$$

respectively. Since these quantities are nonnegative by [A3] and [A4], (2.3) shows that  $v_k^{n+1}$  is a convex combination of  $v_k^n$ ,  $v_{k-1}^n$ , and  $v_{k+1}^n$ . This proves that  $0 \leq v_k^{n+1} \leq M$  for  $k \leq K(n+1)$ . When  $k > K(n+1)$ , these bounds follow from (1.14).

We prove (2.2) first for  $k \leq K(n+1)$ . Rewrite (1.16) as follows:

$$(2.4) \quad v_k^{n+1} = v_k^n + m\beta(v_k^n + \epsilon)(w_{k+1}^n - w_k^n)\Delta x + \frac{m\Delta t}{m-1} \left( \frac{w_{k+1}^n + w_k^n}{2} \right)^2 .$$

We subtract from this the equation corresponding to  $v_{k-1}^{n+1}$  and divide by  $\Delta x$ . Using the discrete product rule

$$\begin{aligned} a_k b_k - a_{k-1} b_{k-1} &= \frac{a_k + a_{k+1}}{2} (b_k - b_{k-1}) \\ &+ \frac{b_k + b_{k-1}}{2} (a_k - a_{k-1}) \quad , \end{aligned}$$

we obtain

$$\begin{aligned} w_k^{n+1} &= w_k^n + m\beta \left( \frac{v_k^n + v_{k-1}^n}{2} + \varepsilon \right) A w_k^n \\ &+ m\beta \Delta x w_k^n \left( \frac{w_{k+1}^n - w_{k-1}^n}{2} \right) \\ &+ \frac{m}{m-1} \beta \Delta x \left( \frac{w_{k+1}^n + 2w_k^n + w_{k-1}^n}{4} \right) (w_{k+1}^n - w_{k-1}^n) \quad , \end{aligned}$$

so that

$$\begin{aligned} (2.5) \quad w_k^{n+1} &= w_k^n + m\beta \left( \frac{v_k^n + v_{k-1}^n}{2} + \varepsilon \right) A w_k^n \\ &+ \frac{m}{m-1} \frac{\beta \Delta x}{4} (w_{k+1}^n + 2mw_k^n + w_{k-1}^n) (w_{k+1}^n - w_{k-1}^n) \quad . \end{aligned}$$

This equation has the form

$$(2.6) \quad w_k^{n+1} = (1 - 2a)w_k^n + (a + b)w_{k+1}^n + (a - b)w_{k-1}^n \quad ,$$

where

$$(2.7) \quad a = m\beta \left( \frac{v_k^n + v_{k-1}^n}{2} + \epsilon \right)$$

and

$$(2.8) \quad b = \frac{m}{m-1} \frac{\beta \Delta x}{4} (w_{k+1}^n + 2mw_k^n + w_{k-1}^n) \quad .$$

By the induction hypotheses (2.1) and (2.2),  $a$  and  $b$  satisfy

$$(2.9) \quad m\beta\epsilon \leq a \leq m\beta(M + \epsilon)$$

and

$$(2.10) \quad |b| \leq \frac{m(m+1)}{m-1} \frac{\beta\gamma_0\Delta x}{2} \quad .$$

Thus, using the mesh conditions [A3] and [A4], we obtain immediately that  $1 - 2a$  and  $a - |b|$  are nonnegative. Hence (2.6) shows that  $w_k^{n+1}$  is a convex combination of  $w_k^n$ ,  $w_{k-1}^n$ , and  $w_{k+1}^n$ , and so satisfies the bound (2.2).

Finally, we prove (2.2) for  $x_k$  near the interface. Thus let  $k = K(n+1)$  and let  $s'_n = \zeta^n - x_k$ , so that

$$w_{k+1}^n = -v_k^n / s'_n \quad .$$

Using the difference equation (1.16), we then have that

$$\begin{aligned}
-w_{k+1}^{n+1} &= \frac{v_k^{n+1}}{s_{n+1}} = \frac{1}{s_{n+1}} [v_k^n + m\beta(v_k^n + \varepsilon)(w_{k+1}^n - w_k^n)\Delta x \\
&\quad + \frac{m\Delta t}{m-1} \{ (w_{k+1}^n)^2 + (\frac{w_{k+1}^n + w_k^n}{2})^2 - (w_{k+1}^n)^2 \}] \\
&= - \frac{w_{k+1}^n}{s_{n+1}} (s_n' - \frac{m}{m-1} w_{k+1}^n \Delta t) \\
&\quad + (w_{k+1}^n - w_k^n) \frac{m}{s_{n+1}} [\beta\Delta x(v_k^n + \varepsilon) - \frac{\Delta t}{m-1} (\frac{3w_{k+1}^n + w_k^n}{4})] .
\end{aligned}$$

But

$$s_{n+1} = s_n' - \frac{m\Delta t}{m-1} w_{k+1}^n$$

by (1.15). Therefore

$$(2.11) \quad w_{k+1}^{n+1} = w_{k+1}^n + c(w_k^n - w_{k+1}^n) ,$$

where

$$(2.12) \quad c = \frac{m}{s_{n+1}} [\beta\Delta x(v_k^n + \varepsilon) - \frac{\Delta t}{m-1} (\frac{3w_{k+1}^n + w_k^n}{4})] .$$

Again, using the induction hypotheses (2.1) and (2.2), we obtain that

$$(2.13) \quad \frac{m\beta\Delta x}{s_{n+1}} (\varepsilon - \frac{\gamma_0\Delta x}{m-1}) \leq c \leq m\beta(M + \varepsilon + \frac{\gamma_0\Delta x}{m-1})$$

the mesh conditions [A3] and [A4] then imply that  $0 \leq c \leq 1$  .

Thus (2.11) shows that  $w_{k+1}^{n+1}$  is a convex combination of  $w_{k+1}^n$  and  $w_k^n$ , and so satisfies the bound (2.2).

Finally, when  $k > K(n+1) + 1$ ,  $w_k^{n+1}$  is between  $w_{k+1}^{n+1}$  and 0, and so again satisfies (2.2). //

The bound (2.2) for  $w_k^n$ , together with the interface condition (1.15), shows that

$$\left| \frac{\zeta^{n+1} - \zeta^n}{\Delta t} \right| \leq C.$$

Since  $\Delta t = O(\Delta x^2)$  by [A4], it follows that  $s_n \leq 2\Delta x + O(\Delta x^2)$  (see (1.10) and (1.12)). Combining this with (1.13), we therefore have

$$(2.14) \quad \Delta x \leq s_n \leq 3\Delta x$$

for small  $\Delta x$ . Actually, any upper bound on  $s_n/\Delta x$  will suffice for our purposes. However, for the sake of simplicity, we shall make use of (2.14) without mentioning the precise conditions on  $\Delta x$  which justify it.

In the next lemma we establish a lower bound for the second spatial differences of  $v_k^n$ . This lower bound will provide a uniform modulus of semicontinuity for  $w_k^n$  and, via (1.15), for  $\frac{\zeta^{n+1} - \zeta^n}{\Delta t}$  as well. This semicontinuity will be crucial later for obtaining error estimates for the approximate interfaces.

Lemma 2.2: Define  $K$  by

$$K = \frac{m-1}{m(m+1)} .$$

Then the bound

$$(2.15) \quad \frac{Av_k^n}{\Delta x^2} = \frac{v_{k+1}^n - 2v_k^n + v_{k-1}^n}{\Delta x^2} \geq - \frac{K}{t_n}$$

holds for all  $k$  and all  $n > 0$  .

Proof: Denote the variable in question by  $z_k^n$  . That is,

$$z_k^n = \frac{Av_k^n}{\Delta x^2} = \frac{w_{k+1}^n - w_k^n}{\Delta x} .$$

Now, if  $0 < t_n \leq \frac{K\Delta x}{2\gamma_0}$  , then

$$|z_k^n| \leq \frac{|w_{k+1}^n| + |w_k^n|}{\Delta x} \leq \frac{2\gamma_0}{\Delta x} \leq \frac{K}{t_n} .$$

We proceed by induction, assuming that (2.15) holds at time level  $n$  , and that

$$(2.16) \quad t_{n+1} > \frac{K\Delta x}{2\gamma_0} .$$

The first case is that in which  $k \leq K(n+1) - 1$  , so that both  $w_k^{n+1}$  and  $w_{k+1}^{n+1}$  satisfy (2.5). Subtracting and dividing by  $\Delta x$  , we thus obtain

$$\begin{aligned}
z_k^{n+1} = & z_k^n + m\beta \left( \frac{v_{k+1}^n + 2v_k^n + v_{k-1}^n}{4} + \epsilon \right) A z_k^n \\
& + m\beta \left( \frac{v_{k+1}^n - v_{k-1}^n}{2\Delta x} \right) \left( \frac{Aw_{k+1}^n + Aw_k^n}{2} \right) \\
& + \frac{m}{m-1} \frac{\beta \Delta x}{8} [w_{k+2}^n + (2m+1)(w_{k+1}^n + w_k^n) + w_{k-1}^n] (z_{k+1}^n - z_{k-1}^n) \\
& + \frac{m}{m-1} \frac{\beta \Delta x}{8} (z_{k+1}^n + 2mz_k^n + z_{k-1}^n) (w_{k+2}^n - w_k^n + w_{k+1}^n - w_{k-1}^n) .
\end{aligned}$$

We rewrite the third term on the right as

$$\frac{m\beta \Delta x}{4} (w_{k+1}^n + w_k^n) (z_{k+1}^n - z_{k-1}^n) ,$$

and the last term on the right as

$$\frac{m}{m-1} \frac{\Delta t}{8} (z_{k+1}^n + 2mz_k^n + z_{k-1}^n) (z_{k+1}^n + 2z_k^n + z_{k-1}^n) .$$

The result is that

$$\begin{aligned}
(2.17) \quad z_k^{n+1} = & (1-2p)z_k^n + (p+q)z_{k+1}^n + (p-q)z_{k-1}^n \\
& + r(z_{k+1}^n + 2mz_k^n + z_{k-1}^n)(z_{k+1}^n + 2z_k^n + z_{k-1}^n) ,
\end{aligned}$$

where

$$p = m\beta \left( \frac{v_{k+1}^n + 2v_k^n + v_{k-1}^n}{4} + \epsilon \right) ,$$

$$q = \frac{m}{m-1} \frac{\beta \Delta x}{8} [w_{k+2}^n + (4m-1)(w_{k+1}^n + w_k^n) + w_{k-1}^n] ,$$

and

$$r = \frac{m}{m-1} \frac{\Delta t}{8} .$$

We shall show that  $Z_k^{n+1}$  in (2.17) is an increasing function of each of the quantities  $Z_k^n$ ,  $Z_{k-1}^n$ , and  $Z_{k+1}^n$ . Using Lemma 2.1, we have

$$\begin{aligned} \frac{\partial Z_k^{n+1}}{\partial Z_k^n} &= (1-2p) + r[(2m+2)(Z_{k+1}^n + Z_{k-1}^n) + 8m Z_k^n] \\ &= (1-2p) + \frac{r}{\Delta x}[(2m+2)(w_{k+2}^n - w_{k-1}^n) + (6m-2)(w_{k+1}^n - w_k^n)] \\ &\geq 1 - 2m\beta(M + \varepsilon) - \frac{m}{m-1} \frac{\Delta t}{8\Delta x} \cdot 16m\gamma_0 \\ &= 1 - 2m\beta[M + \varepsilon + \frac{m}{m-1} \gamma_0 \Delta x] \\ &\geq 0 \end{aligned}$$

by [A4]. And similarly,

$$\begin{aligned} \frac{\partial Z_k^{n+1}}{\partial Z_{k+1}^n} &= p \pm q + 2r [Z_{k+1}^n + (m+1)Z_k^n + Z_{k-1}^n] \\ &= p \pm q + \frac{2r}{\Delta x} [w_{k+2}^n - w_{k-1}^n + m(w_{k+1}^n - w_k^n)] \end{aligned}$$



$$\begin{aligned}
&\geq m\beta\epsilon - \frac{m}{m-1} \frac{\beta\Delta x}{8} \cdot 8m\gamma_0 - \frac{2m}{m-1} \frac{\Delta t}{8\Delta x} \cdot (2m+2)\gamma_0 \\
&= m\beta \left[ \epsilon - \frac{3m+1}{2(m-1)} \gamma_0 \Delta x \right] \\
&\geq 0
\end{aligned}$$

by [A3].

Thus  $z_k^{n+1}$  is bounded below by the right side of (2.17) with  $z_k^n$ ,  $z_{k-1}^n$ , and  $z_{k+1}^n$  replaced by  $-K/t_n$ . That is,

$$\begin{aligned}
z_k^{n+1} &\geq -\frac{K}{t_n} + \frac{m(m+1)}{m-1} \left(\frac{K}{t_n}\right)^2 \Delta t \\
&= -\frac{K}{t_n} \left[ 1 - \frac{m(m+1)}{m-1} \frac{K}{n} \right] \\
&= -\frac{K}{t_n} \frac{n-1}{n} \geq -\frac{K}{t_n} \frac{n}{n+1} = -\frac{K}{t_{n+1}}
\end{aligned}$$

as required.

There are several cases to consider in order to establish the bound (2.15) for  $z_k^{n+1}$  when  $x_k$  is near  $\zeta^{n+1}$ . Now, when  $z_k^{n+1} \geq 0$ , (2.15) is automatically satisfied. We may therefore assume that  $z_k^{n+1} < 0$ ; that is, that

$$\frac{v_{k+1}^{n+1} + v_{k-1}^{n+1}}{2} < v_k^{n+1}.$$

But this shows that  $v_k^{n+1}$  is positive and that  $v_k^{n+1}$  is not computed from the linear interpolation (1.14). Therefore it

must be that  $k \leq K(n+1)$ . Since we already dealt with the case that  $k \leq K(n+1) - 1$ , we may therefore assume that  $k = K(n+1)$ .

Thus  $w_k^{n+1}$  and  $w_{k+1}^{n+1}$  satisfy (2.6) and (2.11) respectively. These equations may be rewritten

$$w_{k+1}^{n+1} = w_{k+1}^n - c\Delta x z_k^n$$

and

$$w_k^{n+1} = w_k^n + (a+b)\Delta x z_k^n - (a-b)\Delta x z_{k-1}^n,$$

where  $a$ ,  $b$ , and  $c$  are as in (2.7), (2.8), and (2.12).

Subtracting and dividing by  $\Delta x$ , we thus obtain

$$(2.18) \quad z_k^{n+1} = (1-a-b-c)z_k^n + (a-b)z_{k-1}^n.$$

We checked in the proof of Lemma 2.1 that  $a-|b| \geq 0$ .

And, using (2.9), (2.10), and (2.13), we have that

$$\begin{aligned} 1-a-b-c &\geq 1 - m\beta(M + \epsilon) - \frac{m(m+1)}{m-1} \frac{\beta\gamma_0\Delta x}{2} - m\beta(M + \epsilon + \frac{\gamma_0\Delta x}{m-1}) \\ &= 1 - 2m\beta[M + \epsilon + \frac{m+3}{4(m-1)} \gamma_0\Delta x], \end{aligned}$$

which is nonnegative by [A4].

Finally, using (2.9), (2.10), (2.13), and (2.14), we have that the sum of the coefficients in (2.18) is

$$1-2b-c \leq 1 + \frac{m(m+1)}{m-1} \beta \gamma_0 \Delta x - \frac{m\beta}{3} \left( \epsilon - \frac{\gamma_0 \Delta x}{m-1} \right) .$$

Using the definition of  $K$ , this bound may be rewritten as

$$1 + \frac{\beta \gamma_0 \Delta x}{K} \left[ 1 + \frac{1}{3(m+1)} \right] - \frac{m\beta \epsilon}{3} .$$

And using condition [A3], we then find that

$$1-2b-c \leq 1 - \frac{2\beta \gamma_0 \Delta x}{K} = 1 - \frac{2\gamma_0}{K} \frac{\Delta t}{\Delta x} .$$

On the other hand, we have from (2.16) that  $t_{n+1} \geq K\Delta x/2\gamma_0$ , so that

$$\frac{1}{n+1} \leq \frac{2\gamma_0}{K} \frac{\Delta t}{\Delta x} .$$

Therefore

$$1-2b-c \leq 1 - \frac{1}{n+1} = \frac{n}{n+1} ,$$

and (2.18) shows that

$$z_k^{n+1} \geq - \frac{K}{t_n} \frac{n}{n+1} = - \frac{K}{t_{n+1}} . \quad //$$

Remark: The constant  $K = (m-1)/m(m+1)$  is the best possible as it can be verified on the Barenblatt-Pattle solution [6,16] (see also [5]).

We can improve the bound (2.15) by imposing additional regularity conditions on  $v_0$ .

Corollary 2.3: a) If  $v_0$  is a concave function, then

$$(2.19) \quad \frac{Av_k^n}{\Delta x^2} \leq 0$$

for all  $k$  and all  $n \geq 0$ .

b) If there is a constant  $C_0$  such that

$$(2.20) \quad \frac{v_0(x+h) - 2v_0(x) + v_0(x-h)}{h^2} \geq -C_0$$

holds for all  $x$  and all  $h > 0$ , then

$$(2.21) \quad \frac{Av_k^n}{\Delta x^2} \geq -C_0$$

for all  $k$  and all  $n \geq 0$ .

Proof: We showed in the proof of Lemma 2.2 that, in all cases,  $Av_k^{n+1}/\Delta x^2$  is an increasing function of  $Av_k^n/\Delta x^2$  and  $Av_{k+1}^n/\Delta x^2$ . The bounds (2.19) and (2.21) then follow easily from (2.17) and (2.18). //

Next, we obtain a bound for the discrete time derivative near the interface.

Lemma 2.4: There is a constant  $C$  such that

$$\left| \frac{v_k^{n+1} - v_k^n}{\Delta t} \right| \leq C$$

holds for  $n \geq 0$  and  $k > K(n+1)$ .

**Proof:** We use the symbol  $O(\cdot)$  to denote dependence on mesh parameters. Let  $k > K \equiv K(m+1)$  and let  $s'_n = \zeta^n - x_K$ . Then from (1.14) we have that

$$\begin{aligned} \frac{v_k^{n+1} - v_k^n}{\Delta t} &= \frac{1}{\Delta t} \left[ \frac{\zeta^{n+1} - x_k}{s_{n+1}} v_K^{n+1} - \frac{\zeta^n - x_k}{s'_n} v_K^n \right] \\ &= O(1) \left[ \frac{v_K^{n+1} - v_K^n}{\Delta t} + \frac{1}{\Delta t} \left( \frac{\zeta^{n+1} - x_k}{s_{n+1}} - \frac{\zeta^n - x_k}{s'_n} \right) v_K^n \right]. \end{aligned}$$

The first term on the right can be estimated by using the difference equation, (1.16):

$$\frac{v_K^{n+1} - v_K^n}{\Delta t} = m(v_K^{n+\varepsilon}) \frac{w_{K+1}^n - w_K^n}{\Delta x} + O(1) = O(1)$$

since  $v_K^{n+\varepsilon} = O(\Delta x)$ . And the second term on the right is

$$O(\Delta x) \cdot \frac{[\zeta^{n+O(\Delta t)} - x_k] s'_n - [s'_n + O(\Delta t)] (\zeta^n - x_k)}{s'_n s_{n+1} \Delta t} = O(1) \quad //$$

Lemmas 2.2 and 2.4, together with the difference equation (1.11), now imply the following bound for the discrete time derivative.

Corollary 2.5: There is a constant  $C$  such that

$$(2.22) \quad \frac{v_k^{n+1} - v_k^n}{\Delta t} \geq -C(1 + \frac{1}{t_n})$$

holds for all  $k$  and all  $n > 0$ .

In the next lemma we use the one-sided bounds (2.15) and (2.22) to derive  $L^1$  estimates for  $Av_k^n/\Delta x^2$  and  $\frac{v_k^{n+1} - v_k^n}{\Delta t}$ .

Lemma 2.6: a) For a given  $T > 0$  there is a constant  $C$  such that

$$\sum_k \left| \frac{Av_k^n}{\Delta x^2} \right| \Delta x, \quad \sum_k \left| \frac{v_k^{n+1} - v_k^n}{\Delta t} \right| \Delta x \leq C(1 + \frac{1}{t_n})$$

holds for  $t_n \leq T$ .

b) If the initial function  $v_0$  satisfies the hypothesis (2.20), then

$$\sum_k \left| \frac{Av_k^n}{\Delta x^2} \right| \Delta x, \quad \sum_k \left| \frac{v_k^{n+1} - v_k^n}{\Delta t} \right| \Delta x \leq C.$$

c) And if  $v_0$  both satisfies (2.20) and is concave, then

$$\left| \frac{Av_k^n}{\Delta x^2} \right|, \quad \left| \frac{v_k^{n+1} - v_k^n}{\Delta t} \right| \leq C.$$

Proof: From Lemma 2.2 we have that

$$\left| \frac{Av_k^n}{\Delta x^2} \right| \leq \frac{Av_k^n}{\Delta x^2} + \frac{2K}{t_n}.$$

We multiply by  $\Delta x$  and sum over  $k$ . Since  $v_k^n$  is zero outside an interval of length  $C(1 + t_n)$ , we obtain that

$$\sum_k \left| \frac{Av_k^n}{\Delta x^2} \right| \Delta x \leq \frac{C}{t_n} (1 + t_n)$$

The other bounds in a) and b) are proved similarly. c) follows from Corollary 2.3, the difference equation (1.11), and Lemma 2.4. //

In the final lemma of this section we establish the Hölder continuity in time of the sequence  $\{v_k^n\}$ .

**Lemma 2.7:** Let  $T > 0$  be given. Then there is a constant  $C$  such that the inequality

$$|v_k^n - v_k^m| \leq C(|t_n - t_m|^{1/2} + \Delta x) \leq (C + \beta^{-1/2})|t_n - t_m|^{1/2}$$

holds for  $t_n$  and  $t_m$  in  $[0, T]$  and for all  $k$ .

Proof: Fix a point  $(x_{k_0}, t_{n_0})$  and let  $t_{n_1} > t_{n_0}$  be given. Let  $Q$  be the rectangle

$$Q = [x_{k_0} - \rho, x_{k_0} + \rho] \times [t_{n_0}, t_{n_1}]$$

where  $\rho$  is a multiple of  $\Delta x$  to be chosen later. And define the quantities

$$H = \max_{n_0 \leq n \leq n_1} |v_{k_0}^n - v_{k_0}^{n_0}| ,$$

$$(2.23) \quad c = 2m(M + \epsilon) + \frac{m}{m-1} \gamma_0 \rho ,$$

and 
$$U_k^n = v_k^n - v_{k_0}^{n_0} - \gamma_0 \rho - \frac{H}{\rho^2} [(x_k - x_{k_0})^2 + c(t_n - t_{n_0})] .$$

We shall show that  $U_k^n \leq 0$  for  $(x_k, t_n) \in Q$  by induction on  $n$ . When  $n = n_0$  and  $|x - x_k| \leq \rho$ , we have, using Lemma 2.1, that

$$U_k^n \leq v_k^{n_0} - v_{k_0}^{n_0} - \gamma_0 \rho \leq 0 .$$

For the induction step we consider the following three cases:

$$|x_k - x_{k_0}| = \rho , \quad k > K(n+1) , \quad \text{and} \quad |x_k - x_{k_0}| < \rho \text{ with } k \leq K(n+1) .$$



In the first of these, we have that

$$U_k^{n+1} \leq (v_k^{n+1} - v_{k_0}^{n+1}) + (v_{k_0}^{n+1} - v_{k_0}^n) - \gamma_0 \rho - H ,$$

which is nonpositive by (2.2) and the definition of  $H$ .

In the second case, we have  $v_k^{n+1} \leq \gamma_0 s_{n+1} \leq 3 \gamma_0 \Delta x$ , so that

$$U_k^{n+1} \leq 3 \gamma_0 \Delta x - \gamma_0 \rho ,$$

which is nonpositive provided that

$$(2.24) \quad \rho \geq 3 \Delta x .$$

For the third case, we employ the linearized difference operator  $L$ , defined, for a given sequence  $z_k^n$ , by

$$\begin{aligned} LZ_k^{n+1} &= \frac{z_k^{n+1} - z_k^n}{\Delta t} - m(v_k^n + \epsilon) \frac{Az_k^n}{\Delta x^2} \\ &\quad - \frac{m}{m-1} \left( \frac{v_{k+1}^n - v_{k-1}^n}{2\Delta x} \right) \left( \frac{z_{k+1}^n - z_{k-1}^n}{2\Delta x} \right) . \end{aligned}$$

Applying  $L$  to  $U_k^n$  and using (1.11), we find that

$$LU_k^{n+1} = \frac{H}{2} [-c + 2m(v_k^n + \epsilon) + \frac{m}{m-1} \left( \frac{v_{k+1}^n - v_{k-1}^n}{2\Delta x} \right) \left( \frac{x_{k+1} - x_{k-1}}{2} - x_{k_0} \right)]$$

so that, by (2.1) and (2.2),

$$LU_k^{n+1} \leq \frac{H}{\rho^2} [-c + 2m(M+\epsilon) + \frac{m}{m-1} \gamma_0 \rho]$$

$$= 0$$

by the definition of  $c$ , (2.23).

On the other hand, we can rewrite the inequality  $LU_k^{n+1} \leq 0$  in the form

$$U_k^{n+1} \leq [1 - 2m\beta(v_k^n + \epsilon)] U_k^n$$

$$+ [m\beta(v_k^n + \epsilon) - \frac{\beta}{4} \frac{m}{m-1} (v_{k+1}^n - v_{k-1}^n)] U_{k+1}^n$$

$$+ [m\beta(v_k^n + \epsilon) + \frac{\beta}{4} \frac{m}{m-1} (v_{k+1}^n - v_{k-1}^n)] U_{k-1}^n .$$

The coefficients of  $U_j^n$  on the right hand side of this inequality are exactly the same as those of  $v_j^n$  in equation (2.3). And we showed in the proof of Lemma 2.1 that these coefficients are nonnegative. We therefore have that  $U_k^{n+1}$  is a convex combination of  $U_{k-1}^n$ ,  $U_k^n$ , and  $U_{k+1}^n$ , and so is nonpositive by the induction hypothesis.

Setting  $k = k_0$  in the result  $U_k^n \leq 0$ , we thus obtain that

$$v_{k_0}^n - v_{k_0}^{n_0} \leq \gamma_0 \rho + \frac{Hc}{\rho^2} s$$

where  $s = t_{n_1} - t_{n_0}$ . In a similar way, we can establish the same inequality for  $v_{k_0}^{n_0} - v_{k_0}^n$ . Taking the maximum over  $n \in [n_0, n_1]$ , we thus obtain

$$(2.25) \quad H \leq \gamma_0 \rho + \frac{Hc}{\rho^2} s.$$

We shall choose  $\rho$  so that  $\frac{cs}{\rho^2} \leq 1/2$ . Specifically,  $\rho$  should satisfy

$$\rho_1 + 3\Delta x \leq \rho \leq \rho_1 + 4\Delta x,$$

where  $\rho_1$  is the larger root of the quadratic equation

$$\rho^2 - 2cs = \rho^2 - \frac{2m}{m-1} \gamma_0 s \rho - 4m(M+\epsilon)s = 0.$$

An easy computation shows that  $\rho_1 = O(s + s^{1/2}) = O(s^{1/2})$  for  $t_{n_i} \leq T$ . And  $\rho \geq 3\Delta x$ , as required by (2.24). Since  $\frac{cs}{\rho^2} \leq 1/2$ , (2.25) becomes

$$H \leq 2\gamma_0 \rho \leq 2\gamma_0(\rho_1 + 4\Delta x) \leq C(s^{1/2} + \Delta x).$$

In particular,

$$|v_{k_0}^{n_1} - v_{k_0}^{n_0}| \leq C(|t_{n_1} - t_{n_0}|^{1/2} + \Delta x). \quad //$$

We remark that the proof of the above lemma is the discrete version of an argument given by Kruzkov, [12] and Gilding [8].

### 3. Convergence of the Approximate Solutions

Let  $h$  denote a pair  $(\Delta x, \Delta t)$  whose elements satisfy the mesh conditions [A3] and [A4]. We define approximate interface curves  $t \mapsto \zeta_r^h(t)$  and  $t \mapsto \zeta_\ell^h(t)$  by piecewise linear interpolation: for  $t_n \leq t \leq t_{n+1}$ ,

$$\zeta_r^h(t) = \zeta_r^n + \frac{m}{m-1} \frac{v_{K(n)}^n}{s_n} (t - t_n),$$

where  $\zeta_r^n$ ,  $v_k^n$ , and  $s_n$  are as in Sections 1 and 2; and similarly for  $\zeta_\ell^h(t)$ . The estimate (2.2) for  $v_{K(n)}^n/s_n$  then shows that the nets  $\{\zeta_r^h(t)\}$  and  $\{\zeta_\ell^h(t)\}$  are uniformly Lipschitz and uniformly bounded in finite time.

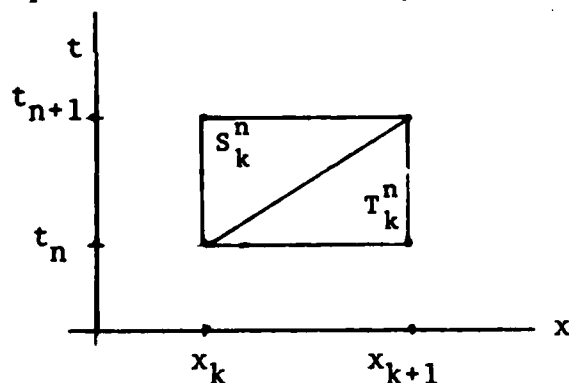


Fig. 3.1

We construct approximate solutions  $v^h(x, t)$  in an analogous way, as follows. If  $T_k^n$  and  $S_k^n$  are the triangles in Fig. 3.1, then

$$v^h(x, t) = v_k^n + (x - x_k)w_{k+1}^n + (t - t_n)\sigma_{k+1}^n, \quad (x, t) \in T_k^n;$$

and

$$v^h(x, t) = v_{k+1}^{n+1} + (x - x_{k+1})w_{k+1}^{n+1} + (t - t_{n+1})\sigma_k^n, \quad (x, t) \in S_k^n.$$

Here

$$w_k^n = \frac{v_k^n - v_{k-1}^n}{\Delta x} \quad \text{and} \quad \sigma_k^n = \frac{v_k^{n+1} - v_k^n}{\Delta t} .$$

It follows immediately from Lemmas 2.1, 2.2, 2.6, and 2.7 that the functions  $v^h$  satisfy

$$(3.1) \quad 0 \leq v^h(x, t) \leq M ,$$

$$(3.2) \quad \left| \frac{\partial v^h}{\partial x} \right| \leq \gamma_0 \quad \text{a.e.} ,$$

$$(3.3) \quad \left| \frac{\partial^2 v^h}{\partial x^2}(x, t) \right| , \quad \left| \frac{\partial v^h}{\partial t}(x, t) \right| \quad \text{for } t > 0 \quad \text{are}$$

finite measures in  $\mathbb{R}$  with mass

$$\int_{\mathbb{R}} \left| \frac{\partial^2 v^h}{\partial x^2} \right| ; \int_{\mathbb{R}} \left| \frac{\partial v^h}{\partial t} \right| \leq C(1 + \frac{1}{t}) .$$

and

$$(3.4) \quad |v^h(x, t+s) - v^h(x, t)| \leq C(T) s^{1/2} , \quad 0 \leq t \leq t+s \leq T .$$

Throughout this section we fix a time  $T$  and a rectangle  $Q = [a, b] \times [0, T]$  , where  $[a, b]$  is large enough to contain the supports of  $v^h(\cdot, t)$  for all  $h$  and all  $t \in [0, T]$  .

(See (1.15) and (2.2) )

The properties of  $\zeta_r^h$  ,  $\zeta_\ell^h$  , and  $v^h$  described above insure that, for every sequence  $h_j$  tending to 0 subject

to the mesh conditions [A3] and [A4], there is a subsequence, which we index simply by  $h$ , for which

$$v^h \longrightarrow v^* \text{ uniformly in } Q ,$$

$$\zeta_r^h, \zeta_\ell^h \longrightarrow \zeta_r^*, \zeta_\ell^* \text{ uniformly in } [0, T] ,$$

and 
$$v_x^h \longrightarrow v_x^* \text{ in } L^2(Q) , \text{ weakly .}$$

Our goal in this section will be to prove that  $v^*$ ,  $\zeta_r^*$ , and  $\zeta_\ell^*$  coincide with the exact solution and interface curves for the problem (1.8). Actually, the convergence of  $v^h$ ,  $\zeta_r^h$ , and  $\zeta_\ell^h$  also follows from the error bounds which we shall derive later in Sections 4 and 5. However, the arguments of the present section are much more direct. Moreover, we obtain here the convergence of  $v_x^h$  in  $L^p(Q)$  for all  $p < \infty$ . As a byproduct of these arguments, we thus obtain in addition a constructive proof of the existence and regularity properties of the solution of (1.8).

We begin by showing that  $v_x^h \rightarrow w^*$  strongly in  $L^p(Q)$  and that, in fact,  $w^* = v_x^*$ .

Lemma 3.1: For any  $t > 0$  the net  $\{v_x^h(\cdot, t)\}$  is pre-compact in  $L^1[a, b]$ .

Proof: The proof consists of estimating the  $L^1$  difference between  $v_x^h(\cdot, t)$  and its spatial translates. Given  $t > 0$ , choose  $n$  so that  $t_n \leq t < t_{n+1}$ . Then when  $h$  is sufficiently small,  $0 < t - \Delta t \leq t_n$ .

If we take  $\rho = \ell \Delta x$  where  $\ell$  is a positive integer, then it is easy to see that

$$\begin{aligned} (3.5) \quad & \int_{\mathbb{R}} |v_x^h(x+\rho, t) - v_x^h(x, t)| dx \\ & \leq C(\sum_k |w_{k+\ell}^n - w_k^n| \Delta x + \sum_k |w_{k+\ell}^{n+1} - w_k^{n+1}| \Delta x) . \end{aligned}$$

The first of these sums is bounded by

$$\begin{aligned} & \sum_k \sum_{j=k+1}^{k+\ell} \frac{|w_j^n - w_{j-1}^n|}{\Delta x} \Delta x^2 \\ & \leq \sum_j \sum_{k=j-\ell}^j \frac{|w_j^n - w_{j-1}^n|}{\Delta x} \Delta x^2 \\ & = \rho \sum_j \frac{|Av_j^n|}{\Delta x^2} \Delta x \leq \frac{K\rho}{t-\Delta t} \end{aligned}$$

by Lemma 2.2. Dealing with the second sum in (3.5) in a similar way, we find that

$$(3.6) \quad \int_{\mathbb{R}} |v_x^h(x+\rho, t) - v_x^h(x, t)| dx \leq \frac{C\rho}{t-\Delta t}$$

holds when  $\rho/\Delta x$  is a positive integer.

When  $\rho_1 < \Delta x$ ,  $|v_x^h(x+\rho_1, t) - v_x^h(x, t)|$  will be zero except when  $x$  is within  $\rho_1$  of the nonhorizontal sides of the triangles  $T_k^n$  and  $S_k^n$ . Thus

$$\begin{aligned} \int_{\mathbb{R}} |v_x^h(x+\rho_1, t) - v_x^h(x, t)| dx \\ \leq C\rho_1 (\sum_k |w_{k+1}^n - w_k^n| + \sum_k |w_{k+1}^{n+1} - w_k^{n+1}|) . \end{aligned}$$

But this expression is  $\frac{\rho_1}{\Delta x}$  times the right side of (3.5) with  $\ell$  taken to be 1. The computations we made above therefore show that

$$\int_{\mathbb{R}} |v_x^h(x+\rho_1, t) - v_x^h(x, t)| dx \leq \frac{C\rho_1}{t - \Delta t} .$$

Combining this with (3.6), we see that (3.6) now holds for all  $\rho > 0$ . The conclusion of the lemma now follows from [1]. //

Lemma 3.1 thus shows that  $\{v_x^h\}$  has strong  $L^1$  limit points. In the next lemma, we prove that these limit points can be identified as the derivatives of limit points of  $\{v^h\}$ .

Lemma 3.2: Let  $v^h$  denote a sequence of approximate solutions which converge to  $v^*$  uniformly in  $Q$ . Then  $v_x^h \rightarrow v_x^*$  in  $L^p(Q)$  for every  $p \in [1, \infty)$ .



Proof: By Lemma 3.1, every subsequence of  $\{v_x^h(\cdot, t)\}$  has a subsequence which converges in  $L^1(\mathbb{R})$ . Thus let  $\{h'\} \subseteq \{h\}$  and let  $v_x^{h'}(\cdot, t)$  converge to a function  $\xi(x)$  in  $L^1(\mathbb{R})$ . We shall show that  $\xi(x) = v_x^*(\cdot, t)$  a.e.

First, if  $\psi \in H_0^1(a, b)$ , then

$$\int_a^b [v_x^{h'}(x, t) - \psi(x, t)] [v_x^{h'}(x, t) - \psi(x, t)] dx = 0.$$

We take

$$\psi = \eta \phi + (1-\eta)v^*(\cdot, t)$$

in this relation, where  $\phi \in H_0^1(a, b)$ . Since  $v^{h'} \rightarrow v^*$  uniformly and  $v_x^{h'}(\cdot, t) \rightarrow \xi$  in  $L^1$  we obtain, by letting  $h \rightarrow 0$ , that

$$\eta \int_a^b [\xi - \eta \phi_x - (1-\eta)v_x^*(\cdot, t)] [\phi - v^*(\cdot, t)] dx = 0.$$

Dividing by  $\eta$  and letting  $\eta \rightarrow 0$ , we then find that

$$\int_a^b [\xi - v_x^*(\cdot, t)] [\phi - v^*(\cdot, t)] dx = 0$$

for all  $\phi \in H_0^1(a, b)$ . This shows that  $\xi = v^*(\cdot, t)$  a.e. Thus every subsequence of  $\{v^h(\cdot, t)\}$  has in turn a subsequence which converges to  $v_x^*(\cdot, t)$  in  $L^1(\mathbb{R})$ . And therefore the entire sequence converges to  $v_x^*(\cdot, t)$  in

$L^1(\mathbb{R})$ . Finally, since  $\|v_x^h\|_\infty \leq \gamma_0$  for every  $h$ , we have that

$$\|v_x^h - v_x^*\|_{p,Q} \leq C(p) \left[ \int_0^T \|v_x^h(\cdot, t) - v_x^*(\cdot, t)\|_{1,\mathbb{R}} dt \right]^{1/p} \\ \rightarrow 0$$

by the dominated convergence theorem. //

We remark that the proof of Lemma 3.2 is an adaptation of an argument given by Minty in [14].

The next theorem contains the main results of this section.

Theorem 3.3: Let  $v$ ,  $\zeta_\ell$ , and  $\zeta_r$  denote the exact solution and interface curves for the problem (1.8). Then

$$(3.7) \quad v^h \rightarrow v \text{ uniformly in } Q,$$

$$(3.8) \quad v_x^h \rightarrow v_x \text{ in } L^p(Q), \quad p < \infty,$$

and

$$(3.9) \quad \zeta_\ell^h, \zeta_r^h \rightarrow \zeta_\ell, \zeta_r \text{ uniformly in } [0, T].$$

We prove (3.7) by showing that limits of converging sequences from  $\{v^h\}$  satisfy the weak equation (1.8), and so agree with its solution  $v$ , which is known to be unique. Thus let  $\{v^h\}$  denote such a sequence and let  $v^h \rightarrow v^*$

uniformly in  $Q$ , so that  $v_x^h(\cdot, t) \rightarrow v_x^*(\cdot, t)$  in  $L^1(\mathbb{R})$  for every  $t > 0$ . It will be sufficient to show that

$$(3.10) \quad \int_{\mathbb{R}} v^*(x, \cdot) \phi(x, \cdot) \Big|_{T_1}^{T_2} dx +$$

$$\int_{T_1}^{T_2} \int_{\mathbb{R}} [-v^* \phi_t + m v^* v_x^* \phi_x + \frac{m(m-2)}{m-1} (v_x^*)^2 \phi] dx dt = 0$$

for all  $C^\infty$  functions  $\phi$  satisfying (1.4) and for  $0 < T_1 < T_2 \leq T$ .

Given such a function  $\phi$ , let  $\phi_k^n = \phi(x_k, t_n)$  and consider the quantity

$$(3.11) \quad \sum_{n=N_1}^{N_2-1} \left\{ \sum_k \left[ \frac{v_k^{n+1} - v_k^n}{\Delta t} - m(v_k^n + \epsilon) \frac{Av_k^n}{\Delta x^2} - \frac{m}{m-1} \left( \frac{v_{k+1}^n - v_{k-1}^n}{2\Delta x} \right)^2 \right] \phi_k^n \Delta x \right\} \Delta t.$$

for appropriate  $N_1$  and  $N_2$ . Now, the expression in brackets vanishes for  $\bar{K}(n+1) \leq k \leq K(n+1)$ . And for other

values of  $k$ , we have that  $\frac{v_k^{n+1} - v_k^n}{\Delta t} = O(1)$  by Lemma 2.4,

and  $(v_k^n + \epsilon) \frac{Av_k^n}{\Delta x^2} = O(\Delta x) \frac{w_{k+1}^n - w_k^n}{\Delta x} = O(1)$  by Lemma 2.1. Thus the quantity (3.11) approaches 0 as  $h \rightarrow 0$ .

On the other hand, we can sum by parts in (3.11) and match the resulting terms with the corresponding integrals in (3.10). We shall carry out the details only for the most complicated term. Using Lemma 2.2, we may rewrite the second two terms in (3.11) as follows:

$$\begin{aligned}
& -m \sum \sum \left[ (v_k^{n+\varepsilon}) \phi_k^n \frac{w_{k+1}^n - w_k^n}{\Delta x} + \frac{1}{m-1} \left( \frac{w_{k+1}^n + w_k^n}{2} \right)^2 \phi_k^n \right] \Delta x \Delta t \\
& = m \sum \sum \left[ w_k^n \left\{ \frac{(v_k^{n+\varepsilon}) \phi_k^n - (v_{k-1}^{n+\varepsilon}) \phi_{k-1}^n}{\Delta x} \right\} - \frac{1}{m-1} (w_k^n)^2 \phi_k^n \right] \Delta x \Delta t + O(\Delta x) \\
(3.12) \quad & = m \sum \sum \left[ w_k^n \left\{ \frac{v_k^n - v_{k-1}^n}{\Delta x} \phi_k^n + (v_{k-1}^{n+\varepsilon}) \frac{\phi_k^n - \phi_{k-1}^n}{\Delta x} \right\} - \frac{1}{m-1} (w_k^n)^2 \phi_k^n \right] \Delta x \Delta t + O(\Delta x) \\
& = \sum \sum \frac{m(m-2)}{m-1} (w_k^n)^2 \phi_k^n \Delta x \Delta t + m \sum \sum v_k^n w_k^n \phi_x(\tilde{x}_k^n, t_n) \Delta x \Delta t + O(\Delta x)
\end{aligned}$$

for some  $\tilde{x}_k^n \in [x_{k-1}, x_k]$ . We shall show that the second sum here converges to the second term on the right in (3.10). First note that, since  $v_x^h(x, t) = w_k^n$  on  $T_k^n \cup S_k^{n-1}$ ,

$$w_k^n \Delta x \Delta t = \iint_{T_k^n \cup S_k^{n-1}} v_x^h(x, t) dx dt.$$

And, since  $\phi$  is smooth and  $v^h$  is Lipschitz in  $x$  and Hölder continuous in  $t$ ,

$$\begin{aligned}
& v_k^n w_k^n \phi_x(\tilde{x}_k^n, t_n) \Delta x \Delta t = \\
& \iint_{T_k^n \cup S_k^{n-1}} v_x^h v_x^h \phi_x dx dt + O(\Delta x + \Delta t^{1/2}) \Delta x \Delta t.
\end{aligned}$$

Therefore the second sum in (3.12) is

$$\int_{T_1}^{T_2} \int_{\mathbb{R}} m v_x^h v_x^h \phi_x dx dt + O(\Delta x),$$

which approaches

$$\int_{T_1}^{T_2} \int_{\mathbb{R}} m v^* v_x^* \phi_x dx dt$$

as  $h \rightarrow 0$ , since  $v^h \rightarrow v^*$  in  $L^\infty$  and  $v_x^h \rightarrow v_x^*$  in  $L^1$ , with  $v^h$  and  $v_x^h$  uniformly bounded. The other terms in (3.11) are handled in a similar manner. Thus  $v^*$  satisfies (3.10) for all appropriate test functions, and so coincides with the unique solution  $v$  of the problem (1.8).

The proof of (3.7) is based upon the following technical lemma, which will be used again in Section 4 for the derivation of error bounds for the approximate interface curves.

Lemma 3.4: Let  $\{\zeta_r^h\}$  be a subsequence such that  $\zeta_r^h \rightarrow \zeta_r^*$  uniformly in  $[0, T]$  as  $h \rightarrow 0$ . Then for every  $t > 0$  and for any positive numbers  $\delta$  and  $\eta$ ,

$$(3.13) \quad \int_t^{t+\eta} v(\zeta_r^*(s) - \delta, s) ds \geq \frac{m-1}{m} \left\{ \frac{\delta}{2} [\zeta_r^*(t+\eta) - \zeta_r^*(t)] - \frac{\delta^2 \eta}{(m+1)t} \right\},$$

for  $0 < t$ ,  $t + \eta \leq T$ . Moreover, if  $\frac{d\zeta_r^*}{dt}$  exists and is positive at  $t$ , then there are positive numbers  $\delta_0$  and  $C$  such that

$$(3.14) \quad v(\zeta_r^*(t) - \delta, t) \geq C\delta$$

holds for  $0 \leq \delta \leq \delta_0$ .

Proof: Let  $p$ ,  $q$ , and  $N$  denote the largest integers in  $\delta/\Delta x$ ,  $n/\Delta t$ , and  $t/\Delta t$  respectively. Then

$$(3.15) \quad \sum_{n=N}^{N+q-1} v_{K(n)-p}^n \Delta t = \sum_{n=N}^{N+q-1} [v_{K(n)}^n - \sum_{j=K(n)-p+1}^{K(n)} w_j^n \Delta x] \Delta t .$$

using Lemma 2.2 and the definition (1.15) of  $\zeta_r^h$ , we have that

$$\begin{aligned} - \sum_{j=K(n)-p+1}^{K(n)} w_j^n \Delta x &= \sum_{j=K(n)-p+1}^{K(n)} [-w_{K(n)}^n + \sum_{\ell=j}^{K(n)-1} \frac{A v_{\ell}^n}{\Delta x^2} \Delta x] \Delta x \\ &\geq \frac{m-1}{m} \sum_{j=K(n)-p+1}^{K(n)} \left[ \frac{\zeta_r^h(t_{n+1}) - \zeta_r^h(t_n)}{\Delta t} - \frac{1}{m+1} \frac{p \Delta x}{t_n} \right] \Delta x \\ &= \frac{m-1}{m} \left[ \frac{\zeta_r^h(t_{n+1}) - \zeta_r^h(t_n)}{\Delta t} p \Delta x - \frac{1}{m+1} \frac{(p \Delta x)^2}{t_n} \right] \Delta x . \end{aligned}$$

Substituting this into (3.15) and discarding the nonnegative term  $v_{K(n)}^n$ , we obtain

$$(3.16) \quad \sum_{n=N}^{N+q-1} v_{K(n)-p}^n \Delta t \geq \frac{m-1}{m} \left\{ [\zeta_r^h(t_{N+q}) - \zeta_r^h(t_N)] p \Delta x - \frac{(p \Delta x)^2 (q \Delta t)}{(m+1) t_N} \right\} .$$

Now (3.13) follows by letting  $h \rightarrow 0$  and using the uniform convergence of  $v^h$  and  $\zeta_r^h$ .

If  $\frac{d\zeta_r^*}{dt}(t) > 0$ , then there is a positive number  $\rho$  such that

$$\zeta_r^*(t+\eta) \geq \zeta_r^*(t) + \rho\eta$$

for small  $\eta$ . For such  $\eta$ , then, (3.13) shows that

$$\int_t^{t+\eta} v(\zeta_r^*(s) - \delta, s) ds \geq \frac{m-1}{m} \left[ \frac{\delta}{2} \rho\eta - \frac{\delta^2 \eta}{(m+1)t} \right].$$

Dividing by  $\eta$  and letting  $\eta \rightarrow 0$ , we thus obtain that

$$\begin{aligned} v(\zeta_r^*(t) - \delta, t) &\geq \frac{m-1}{m} \left[ \frac{\delta\rho}{2} - \frac{\delta^2}{(m+1)t} \right] \\ &\geq \left( \frac{m-1}{m} \frac{\rho}{4} \right) \delta \end{aligned}$$

$$\text{if } \delta \leq \delta_0 \equiv \frac{(m+1)t\rho}{4}. \quad //$$

Proof of 3.9: Let  $\{\zeta_r^h\}$  denote any subsequence converging to a curve  $\zeta_r^*$  uniformly in  $[0, T]$ . We shall show that  $\zeta_r^*(t) = \zeta_r(t)$  for every  $T$ . First observe that, since  $v^h(x, t) = 0$  for  $x \geq \zeta_r^h(t)$ ,  $v(x, t)$  must be 0 for  $x \geq \zeta_r^*(t)$ ; thus  $\zeta_r(t) \leq \zeta_r^*(t)$ .

Now suppose that  $\zeta_r < \zeta_r^*$  on  $(\bar{t}, \bar{t}+\eta)$  with  $\zeta_r(\bar{t}) = \zeta_r^*(\bar{t})$ . Then since  $\zeta_r$  is increasing, there must be a time  $t \in (\bar{t}, \bar{t}+\eta)$  at which  $d\zeta_r^*/dt$  exists and is positive. But then (3.14) shows that  $v(\zeta_r^*(t) - \delta, t)$  is positive for small  $\delta$ . However, this implies that  $\zeta_r(t) \geq \zeta_r^*(t)$ , which is a contradiction. Therefore there is no maximal time  $\bar{t}$  for which  $\zeta_r^*(t) = \zeta_r(t)$  for  $0 \leq t \leq \bar{t}$ . And since  $\zeta_r$  and  $\zeta_r^*$  agree at  $t = 0$ , they agree for all  $t$ . Similar arguments hold for  $\zeta_\ell^h(t)$ . //

#### 4. Error Bounds for the Approximate Solution and Interface Curves

In this section we prove the following theorem.

**Theorem 4.1:** Fix  $T > 0$ . Then there is a constant  $C$  such that, for  $0 \leq t \leq T$ ,

$$(4.1) \quad \|v^h(\cdot, t) - v(\cdot, t)\|_{\infty, \mathbb{R}} \leq C \min \left[ \left( \frac{\Delta x^\alpha |\log \Delta x|}{t} \right)^{\frac{1}{p+3}}, t^{1/2 + \Delta x} \right]$$

and

$$(4.2) \quad |\zeta^h(t) - \zeta(t)| \leq C t^{\frac{1}{m+1}} (\Delta x^\alpha |\log \Delta x|)^{\frac{1}{2(p+3)}}$$

where  $\zeta$  is either  $\zeta_\ell$  or  $\zeta_r$ . Here

$$\alpha = \begin{cases} 1 & , \quad 1 < m < 2 \\ \frac{1}{m-1} & , \quad 2 \leq m \end{cases}$$

and

$$p = \begin{cases} \frac{m+1}{m-1} & , \quad 1 < m < 2 \\ m+1 & , \quad 2 \leq m \end{cases}.$$

We remark that, if the initial data  $v_0$  satisfies the hypothesis (2.20), then the term  $|\log \Delta x|$  may be omitted from the bounds in (4.1) and (4.2).



The proof of Theorem 4.1 will be given in a sequence of lemmas. First we introduce the weak truncation error associated with an approximate solution. If  $\phi$  is a smooth function satisfying (1.4), define

$$(4.3) \quad J(v, \phi, t_1, t_2) = \int_{t_1}^{t_2} \int_{\mathbb{R}} [v_t \phi + m v v_x \phi_x + \frac{m(m-2)}{m-1} v_x^2 \phi] dx dt .$$

Thus  $v$  is a weak solution of (1.6) if and only if  $J(v, \phi, t_1, t_2) = 0$  for all  $\phi$  and all intervals  $(t_1, t_2)$ . The weak truncation error associated with the approximate solution  $v^h$  is then the functional  $J(v^{h+\epsilon}, \cdot, t_1, t_2)$ . We have the following estimate for  $J$ .

Lemma 4.2: Let  $f$  satisfy (1.4) and assume that  $f_t$  and  $f_x$  are in  $L^\infty(\mathbb{R} \times [0, T])$  with  $f(x, T) = 0$  for all  $x$ . Then if  $\phi = (v^{h+\epsilon})^{\frac{2-m}{m-1}} f$ ,

$$(4.4) \quad |J(v^{h+\epsilon}, \phi, \delta, T)| \leq C \|f\| \Delta x^\alpha |\log \delta|$$

where

$$\|f\| = \|f\|_\infty + \|f_t\|_\infty + \|f_x\|_\infty .$$

The proof of Lemma 4.2 is quite technical and lengthy. We therefore postpone it to Section 5.

Next, we define functions  $u$  and  $u^h$  by the relations

$$u = v^{\frac{1}{m-1}} \quad \text{and} \quad u^h = (v^{h+\varepsilon})^{\frac{1}{m-1}}.$$

In the following lemma we exploit the above estimate for  $J$  to obtain a bound for  $u^h - u$  in  $L^{m+1}$ .

Lemma 4.3: There is a constant  $C$  such that

$$\int_0^T \int_a^b |u^h - u|^{m+1} dx dt + \int_0^T \left[ \int_{\mathbb{R}} ((u^h)^m - u^m)_x dt \right]^2 dx \leq$$

(4.5)

$$\leq C [L(u, u^h) \Delta x^\alpha |\log \Delta x| + (\Delta x)^{\frac{m}{m-1}}]$$

where

$$L(u, u^h) = \|(u^h)^m - u^m\|_{\infty, \mathbb{R} \times [0, T]} + \left\| \int_t^T ((u^h)^m - u^m)_x d\tau \right\|_{\infty, \mathbb{R} \times [0, T]}.$$

**Proof:** Let  $\phi$  and  $f$  be as in Lemma 4.2. Then after integrating by parts in (4.3) and substituting, we obtain

$$\begin{aligned} J(v^{h+\varepsilon}, \phi, \delta, T) &= (m-1) \int_{\delta}^T \int_{\mathbb{R}} [u_t^h f + (u^h)_x^m f_x] dx dt \\ &= (m-1) \left[ - \int_{\mathbb{R}} u^h(x, \delta) f(x, \delta) dx \right. \\ &\quad \left. - \int_{\delta}^T \int_{\mathbb{R}} (u^h f_t - (u^h)_x^m f_x) dx dt \right]. \end{aligned}$$

(4.6)

We shall replace  $\delta$  in these integrals by 0. The resulting error in the second integral on the right will be no more than  $C \|f\| \delta$ . And to estimate the change in the first integral, we use the fact that  $u^h$  is Hölder continuous in  $t$ , which may be established as follows. When  $m \in (1, 2]$  we have from Lemma 2.7 that

$$\begin{aligned} |u^h(x, t_2) - u^h(x, t_1)| &\leq C |v^h(x, t_2) - v^h(x, t_1)| \\ &\leq C |t_2 - t_1|^{1/2}. \end{aligned}$$

And when  $m > 2$ ,

$$\begin{aligned} |u^h(x, t_2) - u^h(x, t_1)|^{m-1} &\leq |u^h(x, t_2)^{m-1} - u^h(x, t_1)^{m-1}| \\ &= |v^h(x, t_2) - v^h(x, t_1)| \leq c |t_2 - t_1|^{1/2}. \end{aligned}$$

Thus  $u^h$  is Hölder continuous in  $t$  with exponent  $\alpha/2$ , and

$$\left| \int_{\mathbb{R}} [u^h(x, \delta) f(x, \delta) - u^h(x, 0) f(x, 0)] dx \right| \leq C \|f\| \delta^{\alpha/2}.$$

We thus obtain from (4.6) and Lemma 4.2 that

$$\begin{aligned} \int_{\mathbb{R}} u^h(x, 0) f(x, 0) dx + \int_0^T \int_{\mathbb{R}} [u^h f_t - (u^h)_x^m f_x] dx dt \\ \leq C \|f\| (\Delta x^\alpha |\log \delta| + \delta^{\alpha/2}) \end{aligned}$$

Now subtract from this the weak form of the original equation for  $u$ , (1.3), and take  $\delta = \Delta x^2$ . The result is that

$$(4.7) \quad \int_0^T \int_{\mathbb{R}} [(u^h - u) f_t - ((u^h)^m - u^m)_x f_x] dx dt \leq C \|f\| \Delta x^\alpha |\log \Delta x|.$$

We have used here the fact that

$$\|u(\cdot, 0) - u^h(\cdot, 0)\|_{\infty, \mathbb{R}} \leq C \Delta x^\alpha,$$

which follows directly from the definitions of  $u$  and  $u^h$  and from the Lipschitz continuity of  $v_0$ .

In (4.7) choose

$$f(x, t) = \int_T^t [(u^h)^m - u^m](x, s) ds + (T - t) \epsilon^{\frac{m}{m-1}}$$

It is clear that  $f$  satisfies (1.4) and the conditions of Lemma 4.2. Observe also that

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} (u^h - u) f_t &= \int_0^T \int_{\mathbb{R}} (u^h - u) [(u^h)^m - u^m - \epsilon^{\frac{m}{m-1}}] \\ &\geq \int_0^T \int_a^b |u^h - u|^{m+1} - C \epsilon^{\alpha m}, \end{aligned}$$

and that

$$[(u^h)^m - u^m]_x f_x = \frac{1}{2} \frac{\partial}{\partial t} \left[ \int_0^t ((u^h)^m - u^m)_x(x, s) ds \right]^2, \quad ,$$

so that

$$\int_0^T \int_{\mathbb{R}} [(u^h)^m - u^m]_x f_x = -\frac{1}{2} \int_{\mathbb{R}} \left[ \int_0^T ((u^h)^m - u^m)_x(x, s) ds \right]^2 dx .$$

Now (4.5) follows by making these substitutions into (4.7). //

The next lemma contains the corresponding estimate for the error in  $v^h$ .

Lemma 4.4: There is a constant  $C$  such that

$$\int_0^T \int_a^b |v^h - v|^p dx dt \leq C [L(u, u^h) \Delta x^\alpha |\log \Delta x| + \Delta x^{\alpha m}]$$

where  $L$  is as in Lemma 4.3 and  $\alpha$  and  $p$  are as in Theorem 4.1.

Proof: When  $m \geq 2$ , we have

$$|(v^{h+\varepsilon}) - v| = |(u^h)^{m-1} - u^{m-1}| \leq (m-1) M^{\frac{m-2}{m-1}} |u^h - u| .$$

And if  $1 < m < 2$ , then

$$|(v^{h+\varepsilon}) - v|^{\frac{1}{m-1}} \leq |(v^{h+\varepsilon})^{\frac{1}{m-1}} - v^{\frac{1}{m-1}}| = |u^h - u| .$$

Since  $\varepsilon = O(\Delta x)$ , we have in either case that

$$|v^h - v|^p \leq C(|u^h - u|^{m+1} + \Delta x^p)$$

The conclusion then follows from Lemma 4.3. //

In order to deduce the  $L^\infty$  bound for  $v - v^h$  from the above  $L^p$  bound, we shall require the following interpolation inequality.

Lemma 4.5: Let  $z$  be a continuous function on  $S_T = \mathbb{R} \times [0, T]$  with support  $z(\cdot, t) \subseteq [a, b]$  for  $t \in [0, T]$ . Assume also that

$$a) \quad z_x \in L^\infty(S_T),$$

and

$$b) \quad z_t(\cdot, t) \in L^1(\mathbb{R}) \text{ with}$$

$$\|z_t(\cdot, t)\|_{1, \mathbb{R}} \leq \frac{C_0}{t}.$$

Let  $p \in [1, \infty)$  and  $\ell > 0$  be given. Then there is a constant  $C$  independent of  $\ell$  such that the inequality

$$\|z(\cdot, t)\|_{\infty, \mathbb{R}} \leq \frac{C}{(\ell t)^{\frac{1}{p+3}}} F(z) \|z\|_{p, S_T}^{\frac{p}{p+3}}$$

holds for  $0 < t \leq T - \ell$ . Here

$$F(z) = \|z_x\|_{\infty, S_T}^{\frac{1}{p+1}} \left[ \sup_{t \leq s \leq T} \|z(\cdot, s)\|_{p, \mathbb{R}}^{\frac{2p}{p+1}} + C_0 \|z_x\|_{\infty, S_T}^{\frac{p-1}{p+1}} \right]^{\frac{1}{p+3}}.$$

Proof: For  $t > 0$  we have

$$\begin{aligned} |z(x, t)|^{p+1} &\leq (p+1) \int_{\mathbb{R}} |z(x, t)|^p |z_x(x, t)| dx \\ (4.8) \qquad &\leq C \|z_x\|_{\infty, Q} y(t)^p \end{aligned}$$

where

$$y(t) = \|z(\cdot, t)\|_{p, \mathbb{R}}.$$

Differentiating the definition of  $y(t)$ , we obtain

$$(4.9) \qquad y(t)^{p-1} \left| \frac{dy}{dt} \right| \leq C \|z(\cdot, t)\|_{\infty, \mathbb{R}}^{p-1} \|z_t(\cdot, t)\|_{1, \mathbb{R}}.$$

Combining (4.8) and (4.9), we thus obtain

$$y(t)^{p-1} \left| \frac{dy}{dt} \right| \leq C y(t)^{\frac{p(p-1)}{p+1}} \|z_x\|_{\infty, S_T}^{\frac{p-1}{p+1}} \|z_t(\cdot, t)\|_{1, \mathbb{R}},$$

so that

$$(4.10) \qquad y(t)^{\frac{p-1}{p+1}} \left| \frac{dy}{dt} \right| \leq C \|z_x\|_{\infty, S_T}^{\frac{p-1}{p+1}} \|z_t(\cdot, t)\|_{1, \mathbb{R}}.$$

Now let  $\chi(t)$  be the cut-off function

$$\chi(t) = \begin{cases} 1 & , \quad 0 \leq t \leq T-\ell \\ 1 - \frac{1}{\ell}[t - (T-\ell)] & , \quad T-\ell < t \leq T \end{cases} .$$

Then for  $q > 1$  and  $0 < t \leq T-\ell$ , we have

$$\begin{aligned} y(t)^q &= (y\chi)^q \Big|_T^t \\ &\leq q \int_t^T (\chi y)^{q-1} (|\chi_t| y + \chi \left| \frac{dy}{dt} \right|) \\ &\leq \frac{q}{\ell} \int_t^T (y^q + y^{q-1} \left| \frac{dy}{dt} \right|) \\ &= \frac{q}{\ell} \int_t^T y^{q-1-\frac{p-1}{p+1}} (y^{1+\frac{p-1}{p+1}} + y^{\frac{p-1}{p+1}} \left| \frac{dy}{dt} \right|) . \end{aligned}$$

Now choose

$$q = \frac{p(p+3)}{p+1} > 1$$

and use (4.10) to estimate the second term in the above integrand. The result is that

$$y(t)^{\frac{p(p+3)}{p+1}} \leq \frac{C}{\ell} \int_0^T y(t)^p dt \sup_{t \leq s \leq T} [y(s)^{\frac{2p}{p+1}} + \|z_x\|_{\infty, S_T}^{\frac{p-1}{p+1}} \|z_t(\cdot, s)\|_{1, \mathbb{R}}] .$$



Using hypothesis b), this may be rewritten as

$$(4.11) \quad y(t)^{\frac{p(p+3)}{p+1}} \leq \frac{C}{\ell t} \|z\|_{p, S_T}^p \left[ \sup_{t \leq s \leq T} \|z(\cdot, s)\|_{p, \mathbb{R}}^{\frac{2p}{p+1}} + C_0 \|z_x\|_{\infty, S_T}^{\frac{p-1}{p+1}} \right].$$

Now by (4.8) we have that

$$(4.12) \quad \|z(\cdot, t)\|_{\infty, \mathbb{R}} \leq C \|z_x\|_{\infty, S_T}^{\frac{1}{p+1}} y(t)^{\frac{p}{p+1}}.$$

The result then follows by substituting (4.11) into (4.12). //

We can now prove the first half of Theorem 4.1 by applying the interpolation inequality of Lemma 4.5 to the  $L^p$  error bound in Lemma 4.4 as follows.

Proof of (4.1): It is clear that  $v^h - v$  satisfies the first hypothesis of Lemma 4.5. In addition, hypothesis b) is satisfied because of (3.3) and the results of [5]. (Alternatively, we can observe that the conclusion of Lemma 4.5 remains valid for limits of functions which satisfy hypotheses a) and b) uniformly. And  $v$  is such a function by the results of Section 3.) Replacing  $T-\ell$  and  $T$  by  $T$  and  $T+1$  respectively, we therefore obtain that

$$\|v^h(\cdot, t) - v(\cdot, t)\|_{\infty, \mathbb{R}} \leq \frac{c}{t^{\frac{1}{p+3}}} \|v^h - v\|_{p, S_T}^{\frac{p}{p+3}}$$

where  $S_T^1 = \mathbb{R} \times [0, T+1]$  and  $p \in [1, \infty)$ . But if we choose  $p$  as in the statement of Theorem 4.1, then Lemma 4.4 shows that

$$\|v^h - v\|_{p, S_T^1}^p \leq C \Delta x^\alpha |\log \Delta x|.$$

Therefore

$$\|v^h(\cdot, t) - v(\cdot, t)\|_{\infty, \mathbb{R}} \leq C \left( \frac{\Delta x^\alpha |\log \Delta x|}{t} \right)^{\frac{1}{p+3}}.$$

On the other hand, the Hölder continuity of  $v^h$  and  $v$  in time shows that

$$\|v^h(\cdot, t) - v(\cdot, t)\|_{\infty, \mathbb{R}} \leq C (t^{1/2} + \Delta x)$$

for any  $t$ . The estimate (4.1) follows from these last two inequalities. //

Next, we deduce the bound (4.2) for the error in the approximate interfaces from the above bound (4.1) for the error in  $v^h$ . Again we drop the subscripts and denote  $\zeta_r^h$  and  $\zeta_r$  by  $\zeta^h$  and  $\zeta$ .

Proof of (4.2): First we refine the result in Lemma 3.4. Divide (3.13) by  $\eta$  and let  $\eta \rightarrow 0$ . The result is that

$$(4.13) \quad v(\zeta(t)-y, t) \geq \frac{m-1}{m} \left[ \frac{y}{2} \dot{\zeta}(t) - \frac{y^2}{(m+1)t} \right] \text{ a.e.}$$

A similar bound holds for  $v^h$ : taking  $q = 1$  in (3.16), we have that

$$v_{K(n)-j}^n \geq \frac{m-1}{m} \left[ (j\Delta x) \dot{\zeta}^h(t_n) - \frac{(j\Delta x)^2}{(m+1)t_n} \right].$$

Since  $v^h$  is Lipschitz in  $x$  and Hölder continuous in  $t$ , and since  $\zeta^h$  is piecewise linear, we can conclude that

$$(4.14) \quad v^h(\zeta^h(t)-y, t) \geq \frac{m-1}{m} \left[ \dot{\zeta}^h(t)y - \frac{y^2}{(m+1)t} \right] - C\Delta x \text{ a.e.}$$

Now let  $\{v_1, v_2\} = \{v, v^h\}$  and let  $\zeta_1$  and  $\zeta_2$  be the corresponding interface curves. Assume that, at time  $t$ ,  $0 \leq \zeta_2(t) - \zeta_1(t) = y$ . Then we have, using either (4.13) or (4.14), that

$$\begin{aligned} (4.15) \quad \|v^h(\cdot, t) - v(\cdot, t)\|_{\infty, \mathbb{R}} &\geq |(v_2 - v_1)(\zeta_1(t), t)| \\ &= v_2(\zeta_1(t), t) = v_2(\zeta_2(t) - y, t) \\ &\geq \frac{m-1}{m} \left[ \dot{\zeta}_2(t)y - \frac{y^2}{(m+1)t} \right] - C\Delta x \\ &\geq \frac{m-1}{m} \left[ (\dot{\zeta}_2 - \dot{\zeta}_1)(\zeta_2 - \zeta_1) - \frac{(\zeta_2 - \zeta_1)^2}{(m+1)t} \right] - C\Delta x. \end{aligned}$$

Now let  $F(t) = [\zeta_2(t) - \zeta_1(t)]^2$  and let  $E(t)$  be the bound in (4.1) for  $\|v^h(\cdot, t) - v(\cdot, t)\|_{\infty, \mathbb{R}}$ . (4.15) then shows that

$$\dot{F}(t) - \frac{2}{(m+1)t} F(t) \leq E(t) \quad ,$$

where we have subsumed the  $C\Delta x$  term into  $E(t)$ . Integrating, we thus obtain that, for any  $\delta > 0$ ,

$$(4.16) \quad F(t) \leq t^{\frac{2}{m+1}} \left[ \delta^{-\frac{2}{m+1}} F(\delta) + \int_{\delta}^T s^{-\frac{2}{m+1}} E(s) ds \right] \quad .$$

Now, since

$$\begin{aligned} F(\delta) &= [\zeta^h(\delta) - \zeta(\delta)]^2 \\ &\leq (|\zeta^h(\delta) - \zeta^h(0)| + |\zeta(0) - \zeta(\delta)|)^2 \\ &\leq C\delta^2 \quad , \end{aligned}$$

the first term on the right of (4.16) approaches 0 as  $\delta \rightarrow 0$ . In addition, the integrand in the second term of (4.16) is bounded by

$$C s^{-(\frac{1}{m+1} + \frac{1}{p+3})} (\Delta x^\alpha |\log \Delta x|)^{\frac{1}{p+3}}$$

A short computation shows that the exponent of  $s$  here is greater than  $-1$ , so that this term is integrable on  $[0, t]$ .

We may therefore conclude that

$$|\zeta^h(t) - \zeta(t)| = F(t)^{1/2} \leq C t^{\frac{1}{m+1}} (\Delta x^\alpha |\log \Delta x|)^{\frac{1}{2(p+3)}}. //$$

### 5. Estimate for the weak truncation error

This section is devoted to the proof of Lemma 4.2. Since the arguments are technically involved, even though simple, we begin by listing a few facts to which we will refer systematically. In the estimates to follow we will use repeatedly the bounds established in Section 2 without specific mention.

Let  $T_k^n$  and  $S_k^n$  be the triangles in Fig. 3.1, and let  $v^h$  be the piecewise linear interpolated of  $v_k^n$  introduced in Section 3. In  $T_k^n$ ,  $v^h$  can be written in any one of the following equivalent forms

$$(5.1) \quad v^h(x, t) \equiv \begin{cases} v_k^n + w_{k+1}^n(x - x_k) + \sigma_{k+1}^n(t - t_n) \\ v_{k+1}^n + w_{k+1}^n(x - x_{k+1}) + \sigma_{k+1}^n(t - t_n) \\ v_{k+1}^{n+1} + w_{k+1}^n(x - x_{k+1}) + \sigma_{k+1}^n(t - t_{n+1}) \end{cases},$$

and in  $S_k^n$  it can be written in any one of the equivalent forms

$$(5.2) \quad v^h(x, t) = \begin{cases} v_k^n + w_{k+1}^{n+1}(x - x_k) + \sigma_k^n(t - t_n) \\ v_k^{n+1} + w_{k+1}^{n+1}(x - x_k) + \sigma_k^n(t - t_{n+1}) \\ v_{k+1}^{n+1} + w_{k+1}^{n+1}(x - x_{k+1}) + \sigma_k^n(t - t_{n+1}) \end{cases}.$$

Let  $\phi \in H^1(S_T)$  be a test function satisfying (1.4), denote with  $\phi^h$  the piecewise linear interpolation of the values  $\phi_k^n = \phi(x_k, t_n)$  and set

$$\psi_k^n = \frac{\phi_k^n - \phi_{k-1}^n}{\Delta x}, \quad \phi_k^n = \frac{\phi_k^{n+1} - \phi_k^n}{\Delta t}.$$

In  $T_k^n$ ,  $\phi^h$  can be written in any of the equivalent forms

$$(5.3) \quad \phi^h(x, t) = \begin{cases} \phi_k^n + \psi_{k+1}^n(x - x_k) + \phi_{k+1}^n(t - t_n) \\ \phi_{k+1}^n + \psi_{k+1}^n(x - x_{k+1}) + \phi_{k+1}^n(t - t_n) \\ \phi_{k+1}^{n+1} + \psi_{k+1}^n(x - x_{k+1}) + \phi_{k+1}^n(t - t_{n+1}) \end{cases}$$

and in  $S_k^n$

$$(5.4) \quad \phi^h(x, t) = \begin{cases} \phi_k^n + \psi_k^{n+1}(x - x_k) + \phi_k^n(t - t_n) \\ \phi_k^{n+1} + \psi_k^{n+1}(x - x_k) + \phi_k^n(t - t_{n+1}) \\ \phi_{k+1}^{n+1} + \psi_k^{n+1}(x - x_{k+1}) + \phi_k^n(t - t_{n+1}) \end{cases}$$

**Remark** From (5.1)-(5.2) it follows that  $v^h$  can be written as the value of  $v^h$  at any one of the corners of  $T_k^n$  ( $S_k^n$  respectively), plus terms of the order of  $\Delta x$ . An analogous fact holds for  $\phi^h$ .

### 5.1 The basic identity

Consider the quantity

$$(5.5) \quad J(v^h + \varepsilon, \phi^h, \delta, T) =$$

$$= \iint_{S_{T,\delta}} \{v_t^h \phi^h + m(v^h + \varepsilon) v_x^h \phi_x^h + \frac{m(m-2)}{m-1} (v_x^h)^2 \phi^h\} dx dt$$

where  $\delta$  is a fixed positive number. We assume for simplicity that  $T = (N+1)\Delta t$  and  $\delta = n_0 \Delta t$  for two positive integers  $n_0 < N$ , and calculate the various parts of (5.5) as follows.

$$(i) \quad J_1 = \iint_{S_{T,\delta}} v_t^h \phi^h dx d\tau = \sum_{n_0}^N \sum_Z \{ \iint_{T_k^n} v_t^h \phi^h dx d\tau + \iint_{S_k^n} v_t^h \phi^h dx dt \}$$

$$= \sum_{n_0}^N \sum_Z \sigma_k^n \phi_k^n \Delta x \Delta t + \sum_{n_0}^N \sum_Z \frac{1}{6} [\sigma_k^n \psi_{k+1}^{n+1} - \sigma_k^n \psi_k^n] (\Delta x)^2 \Delta t .$$

$$(ii) \quad J_2 = m \iint_{S_{T,\delta}} \{ (v^h + \varepsilon) v_x^h \phi_x^h + (v_x^h)^2 \phi^h \} dx d\tau =$$

$$= m \iint_{S_{T,\delta}} v_x^h [(v^h + \varepsilon) \phi^h]_x dx d\tau =$$

$$= \sum_{n_0}^N \sum_Z \{ \iint_{T_{k+1}^n} w_{k+1}^n [(v^h + \varepsilon) \phi^h]_x dx d\tau + \iint_{S_k^n} w_{k+1}^{n+1} [(v^h + \varepsilon) \phi^h]_x dx d\tau \} .$$



On  $T_k^n$  we have

$$\begin{aligned} [(v_k^{n+\varepsilon}) \phi^h]_x &= (v_{k+1}^{n+\varepsilon}) \psi_{k+1}^n + w_{k+1}^n \phi_{k+1}^n + \\ &+ w_{k+1}^n \psi_{k+1}^n [(x-x_k) + (x-x_{k+1})] + \\ &+ (\sigma_{k+1}^n \psi_{k+1}^n + w_{k+1}^n \phi_{k+1}^n) (t-t_n) . \end{aligned}$$

Therefore

$$\begin{aligned} \iint_{T_k^n} w_{k+1}^n [(v_k^{n+\varepsilon}) \phi^h]_x dx d\tau &= \frac{1}{2} [(v_{k+1}^{n+\varepsilon}) \psi_{k+1}^n + w_{k+1}^n \phi_{k+1}^n] \Delta x \Delta t + \\ &+ \frac{1}{6} [\sigma_{k+1}^n \psi_{k+1}^n + w_{k+1}^n \phi_{k+1}^n] w_{k+1}^n \Delta x (\Delta t)^2 . \end{aligned}$$

By direct calculation

$$\begin{aligned} w_{k+1}^n [(v_k^{n+\varepsilon}) \psi_{k+1}^n + w_{k+1}^n \phi_{k+1}^n] &= -(v_k^{n+\varepsilon}) \frac{[Av]_k^n}{(\Delta x)^2} \phi_k^n + \\ &+ (v_k^{n+\varepsilon}) \frac{w_{k+1}^n - w_k^n}{\Delta x} \phi_{k+1}^n + (v_k^{n+\varepsilon}) w_k^n \frac{\phi_{k+1}^n - \phi_k^n}{\Delta x} + (w_{k+1}^n)^2 \phi_{k+1}^n = \\ &= -(v_k^{n+\varepsilon}) \frac{[Av]_k^n}{(\Delta x)^2} \phi_k^n + \frac{1}{\Delta x} \{ (v_{k+1}^{n+\varepsilon}) w_{k+1}^n \phi_{k+1}^n - (v_k^{n+\varepsilon}) w_k^n \phi_k^n \} . \end{aligned}$$

This implies that

$$\begin{aligned}
& \sum_{n_0}^N \sum_{\mathbf{z}} \iint_{T_k^n} w_{k+1}^n [(v_k^h + \varepsilon) \phi^h]_x dx dt = \\
& = -\frac{1}{2} \sum_{n_0}^N \sum_{\mathbf{z}} (v_k^{n+\varepsilon}) \frac{[Av]_k^n}{(\Delta x)^2} \phi_k^n \Delta x \Delta t + \\
& + \frac{1}{6} \sum_{n_0}^N \sum_{\mathbf{z}} w_k^n [\sigma_k^n \psi_k^n + w_k^n \phi_k^n] \Delta x (\Delta t)^2 .
\end{aligned}$$

Analogous calculations on  $S_k^n$  give

$$\begin{aligned}
& \sum_{n_0}^N \sum_{\mathbf{z}} \iint_{S_k^n} w_{k+1}^{n+1} [(v_k^h + \varepsilon) \phi^h]_x dx dt = \\
& = -\frac{1}{2} \sum_{n_0}^N \sum_{\mathbf{z}} (v_k^{n+1+\varepsilon}) \frac{[Av]_k^{n+1}}{(\Delta x)^2} \phi_k^{n+1} - \\
& - \frac{1}{6} \sum_{n_0}^N \sum_{\mathbf{z}} w_{k+1}^{n+1} [\sigma_k^n \psi_{k+1}^{n+1} + w_{k+1}^{n+1} \phi_k^n] \Delta x (\Delta t)^2
\end{aligned}$$

Substituting these calculations in the expression of  $J_2$  we obtain

$$(ii)' \quad J_2 = \sum_{n_0+1}^N \sum_{\mathbf{z}} m(v_k^{n+\varepsilon}) \frac{[Av]_k^n}{(\Delta x)^2} \phi_k^n -$$

$$\begin{aligned}
& - \frac{m}{2} \Delta t \sum_{\mathbf{Z}} \Delta x \{ (v_k^{n_0+\epsilon}) \frac{[Av]_k^{n_0}}{(\Delta x)^2} \phi_k^{n_0} + (v_k^{N+1+\epsilon}) \frac{[Av]_k^{N+1}}{(\Delta x)^2} \phi_k^{N+1} \} \\
& + \frac{m}{6} \sum_{n_0}^N \sum_{\mathbf{Z}} w_k^n (\sigma_k^n \psi_k^n + w_k^n \phi_k^n) \Delta x (\Delta t)^2 - \\
& - \frac{m}{6} \sum_{n_0}^N \sum_{\mathbf{Z}} w_{k+1}^{n+1} (\sigma_k^n \psi_{k+1}^{n+1} + w_{k+1}^{n+1} \phi_k^n) \Delta x (\Delta t)^2 .
\end{aligned}$$

We finally transform the remaining integral in  $J(v^{h+\epsilon}, \phi^h, \delta, T)$  .

$$\begin{aligned}
\text{(iii)} \quad J_3 &= - \frac{m}{m-1} \iint_{S_{T,\delta}} (v_{\wedge}^h)^2 \phi^h dx dt = \\
&= - \frac{m}{m-1} \sum_{n_0}^N \sum_{\mathbf{Z}} \{ \iint_{T_k^n} (w_{k+1}^n)^2 \phi^h dx dt + \iint_{S_k^n} (w_{k+1}^{n+1})^2 \phi^h dx dt \} .
\end{aligned}$$

Using (5.3)-(5.4), by standard calculations we obtain

$$\begin{aligned}
J_3 &= - \frac{m}{m-1} \sum_{n_0+1}^N \sum_{\mathbf{Z}} (w_k^n)^2 \phi_k^n \Delta x \Delta t + \\
&+ \frac{m}{3(m-1)} \sum_{n_0+1}^N \sum_{\mathbf{Z}} (w_k^n)^2 \psi_k^n (\Delta x)^2 \Delta t - \\
&- \frac{m}{6(m-1)} \sum_{n_0}^N \sum_{\mathbf{Z}} [(w_k^{n+1})^2 - (w_k^n)^2] \phi_k^n \Delta x (\Delta t)^2 -
\end{aligned}$$

$$\begin{aligned}
& - \frac{m}{m-1} \Delta t \sum_{\mathbf{Z}} \left\{ \frac{1}{2} [(w_k^{n_0})^2 \phi_k^{n_0} + (w_k^{N+1})^2 \phi_k^{N+1}] - \right. \\
& \left. - \frac{1}{6} [(w_k^{n_0})^2 \psi_k^{n_0} + (w_k^{N+1})^2 \psi_k^{N+1}] \right\} \Delta x .
\end{aligned}$$

We rewrite the first summand in  $J_3$  as follows

$$(w_k^n)^2 \phi_k^n = \left( \frac{w_{k+1}^n + w_k^n}{2} \right)^2 \phi_k^n + [(w_k^n)^2 - \left( \frac{w_{k+1}^n + w_k^n}{2} \right)^2] \phi_k^n ,$$

and combine the expressions of  $J_i$ ,  $i = 1, 2, 3$  so obtained, as parts of  $J(v^{h+\epsilon}, \phi^h, \delta, T)$  to deduce the following basic identity

$$\begin{aligned}
(5.6) \quad J(v^{h+\epsilon}, \phi^h, \delta, T) &= \\
&= \sum_{n_0+1}^N \sum_{\mathbf{Z}} \left\{ \frac{v_k^{n+1} - v_k^n}{\Delta t} - m(v_k^{n+\epsilon}) \frac{[Av]_k^n}{(\Delta x)^2} - \frac{m}{m-1} \left( \frac{v_{k+1}^n - v_{k-1}^n}{2\Delta x} \right)^2 \right\} \phi_k^n \Delta x \Delta t \\
&+ \Delta t \sum_{\mathbf{Z}} \sigma_k^{n_0} \phi_k^n \Delta x - \\
&- \frac{m\Delta t}{2} \sum_{\mathbf{Z}} \left\{ (v_k^{n_0+\epsilon}) \frac{[Av]_k^{n_0}}{(\Delta x)^2} \phi_k^{n_0} + (v_k^{N+1+\epsilon}) \frac{[Av]_k^{N+1}}{(\Delta x)^2} \phi_k^{N+1} \right\} \Delta x + \\
&+ \frac{\Delta x}{6} \sum_{n_0}^N \sum_{\mathbf{Z}} [\sigma_k^n \psi_{k+1}^{n+1} - \sigma_k^n \psi_k^n] \Delta x \Delta t -
\end{aligned}$$

$$\begin{aligned}
& - \frac{m\Delta t}{6} \sum_{n_0}^N \sum_Z w_{k+1}^{n+1} [\sigma_k^n \psi_{k+1}^{n+1} + w_{k+1}^{n+1} \phi_k^n] \Delta x \Delta t + \\
& + \frac{m\Delta x}{3(m-1)} \sum_{n_0}^N \sum_Z (w_k^n)^2 \psi_k^n \Delta x \Delta t + \\
& + \frac{m\Delta t}{6} \sum_{n_0}^N \sum_Z w_k^n [\sigma_k^n \psi_k^n + w_k^n \phi_k^n] \Delta x \Delta t - \\
& - \frac{m\Delta t}{6(m-1)} \sum_{n_0}^N \sum_Z [(w_k^{n+1})^2 - (w_k^n)^2] \phi_k^n \Delta x \Delta t - \\
& - \frac{m\Delta t}{2(m-1)} \sum_Z [ (w_k^{n_0})^2 \phi_k^{n_0} + (w_k^{N+1})^2 \phi_k^{N+1} ] - \frac{1}{3} [ (w_k^{n_0})^2 \psi_k^{n_0} + (w_k^{N+1})^2 \psi_k^{N+1} ] \Delta x + \\
& + \frac{m\Delta x}{4(m-1)} \sum_{n_0+1}^N \sum_Z \frac{[Av]_k^n}{(\Delta x)^2} [3w_k^n + w_{k+1}^n] \phi_k^n \Delta x \Delta t = \\
& = \sum_{i=1}^{10} H_i .
\end{aligned}$$

From now on we will select test functions  $\phi$  of the form

$$\phi = (v^{h+\varepsilon})^{\frac{2-m}{m-1}} f, \quad m > 1$$

where  $f$  satisfies (1.4). We will estimate the  $H_i$ ,

$i = 1, 2, \dots, 10$  in (5.6) in terms of  $f$ ,  $f_x$ ,  $f_t$ . For

notational simplicity set

$$\|f, f_x, f_t\|_{\infty, S_T} \equiv \|f, f_x, f_t\|$$

and

$$\|f\| = \|f\| + \|f_x\| + \|f_t\|.$$

We start by making elementary estimates of  $\phi$  and its first derivatives in terms of  $f$ .

Denoting by  $\phi^h$  the piecewise linear interpolated of  $\phi$  and recalling that  $\varepsilon$  is of the order of  $\Delta x$  we have

$$(5.7) \quad \|\phi^h\|_{\infty} \leq \begin{cases} C \|f\| & \text{if } 1 < m \leq 2 \\ C \|f\| (\Delta x)^{\frac{2-m}{m-1}} & \text{if } m > 2 \end{cases}.$$

Also

$$\begin{aligned} (a) \quad \psi_k^n &= \frac{\phi_k^n - \phi_{k-1}^n}{\Delta x} = \frac{1}{\Delta x} \left\{ (v_k^n + \varepsilon)^{\frac{2-m}{m-1}} f_k^n - (v_{k-1}^n + \varepsilon)^{\frac{2-m}{m-1}} f_{k-1}^n \right\} = \\ &= \frac{1}{\Delta x} \left\{ [(v_k^n + \varepsilon)^{\frac{2-m}{m-1}} - (v_{k-1}^n + \varepsilon)^{\frac{2-m}{m-1}}] f_k^n + (v_{k-1}^n + \varepsilon)^{\frac{2-m}{m-1}} \frac{f_k^n - f_{k-1}^n}{\Delta x} \right\} \leq \\ &\leq \left| \frac{2-m}{m-1} \right| (v_k^n + \varepsilon)^{\frac{2-m}{m-1} - 1} |w_k^n| \|f\| + (v_{k-1}^n + \varepsilon)^{\frac{2-m}{m-1}} \|f_x\|; \end{aligned}$$

$$(b) \quad \phi_k^n = \frac{\phi_k^{n+1} - \phi_k^n}{\Delta t} \leq \left| \frac{2-m}{m-1} \right| (v_k^{\tilde{n}+\epsilon})^{\frac{2-m}{m-1}-1} |\sigma_k^n| \|f\| + (v_k^{\tilde{n}+\epsilon})^{\frac{2-m}{m-1}} \|f_t\| ,$$

where

$$v_k^{\tilde{n}} = \min \{v_k^n; v_{k-1}^n\}$$

$$v_k^{\tilde{n}} = \min \{v_k^{n+1}, v_k^n\} .$$

From (a)-(b) we deduce

$$(5.8) \quad |\psi_k^n| \leq \begin{cases} c(v_k^{\tilde{n}+\epsilon})^{-1} \|f\| + C \|f_x\| & \text{if } 1 < m \leq 2 \\ C[(v_k^{\tilde{n}+\epsilon})^{-1} \|f\| + \|f_x\|](\Delta x)^{\frac{2-m}{m-1}}, & \text{if } m > 2 \end{cases}$$

$$(5.9) \quad |\phi_k^n| \leq \begin{cases} C(v_k^{\tilde{n}+\epsilon})^{-1} |\sigma_k^n| \|f\| + C \|f_t\| & \text{if } 1 < m \leq 2 \\ C[(v_k^{\tilde{n}+\epsilon})^{-1} |\sigma_k^n| \|f\| + \|f_t\|](\Delta x)^{\frac{2-m}{m-1}} & \text{if } m > 2 \end{cases} .$$

## 5.2 An auxiliary lemma

Lemma 5.1: For all test functions  $\phi$  of the form

$$\phi = (v^{h+\epsilon})^{\frac{2-m}{m-1}} f ,$$

the following estimate holds

$$|J(v^{h+\varepsilon}, \phi^h, \delta, T)| \leq C \|f\| (\Delta x)^\alpha |\log \delta| ,$$

where

$$\alpha = \min \{1; \frac{1}{m-1}\} .$$

Proof: We estimate the  $H_1$  on the right side of (5.6) separately. At points where the difference equation (1.11) holds, the summand in  $H_1$  vanishes, and by virtue of Lemma 2.4, the sum extended over the remaining  $(n, k)$  is of the order of  $\Delta x$ .

Therefore

$$|H_1| \leq C (\Delta x) \|\phi\|_\infty \leq C \|f\| (\Delta x)^\alpha .$$

We have also easily

$$|H_2| + |H_3| \leq C \|f\| (\Delta x)^\alpha .$$

In estimating  $H_4$  we only consider the term whose summand is  $\sigma_k^n \psi_{k+1}^{n+1}$ . The estimate for the term whose summand is  $\sigma_k^n \psi_k^n$  is analogous, and in fact simpler.

We have

$$\Delta x \sum_{n_0}^N \sum_Z \sigma_k^n \psi_{k+1}^{n+1} \Delta x \Delta t = \Delta x \sum_{n_0}^N \sum_{[k < K(n+1)] \cup [k > K(n+1)]} \sigma_k^n \psi_{k+1}^{n+1} \Delta x \Delta t$$



$$+ \Delta x \sum_{n_0}^N \sum_{\bar{K}(n+1) \leq k \leq K(n+1)} \sigma_k^n \psi_{k+1}^{n+1} \Delta x \Delta t = \kappa^{(1)} + \kappa^{(2)}.$$

As for  $\kappa^{(1)}$ , using Lemma 2.4 and (5.8)

$$|\kappa^{(1)}| \leq C(\Delta x)^2 \max_{n,k} |\psi_k^n| \leq C \|f\| (\Delta x)^\alpha.$$

We estimate  $\kappa^{(2)}$  by using the difference equation (1.11).

$$\begin{aligned} \kappa^{(2)} &= \Delta x \sum_{n_0}^N \sum_{\bar{K}(n+1) \leq k \leq K(n+1)} \left[ m(v_{k+\varepsilon}^n) \frac{[Av]_k^n}{(\Delta x)^2} + \frac{m}{m-1} \left( \frac{w_{k+1}^n + w_k^n}{2} \right)^2 \right] \psi_{k+1}^{n+1} \Delta x \Delta t \\ &\leq \Delta x \sum_{n_0}^N \sum_{\bar{K}(n+1) \leq k \leq K(n+1)} m(v_{k+\varepsilon}^n) \left| \frac{[Av]_k^n}{(\Delta x)^2} \right| |\psi_{k+1}^{n+1}| \Delta x \Delta t + \\ &+ \Delta x \left| \sum_{n_0}^N \sum_{\bar{K}(n+1) \leq k \leq K(n+1)} \frac{m}{m-1} \left( \frac{w_{k+1}^n + w_k^n}{2} \right)^2 \psi_{k+1}^{n+1} \Delta x \Delta t \right| = \\ &= \kappa_a^{(2)} + \kappa_b^{(2)}. \end{aligned}$$

Estimate the summand in  $\kappa_a^{(2)}$  as follows

$$m \Delta x (v_{k+\varepsilon}^n) \left| \frac{[Av]_k^n}{(\Delta x)^2} \right| |\psi_{k+1}^{n+1}| \leq C \|f_x\| (\Delta x)^\alpha \left| \frac{[Av]_k^n}{(\Delta x)^2} \right| +$$

$$+ C \left( \frac{v_k^{n+\varepsilon}}{v_{\tilde{k}}^{n+1+\varepsilon}} \right) \|f\| (\Delta x)^\alpha \left| \frac{[Av]_k^n}{(\Delta x)^2} \right| .$$

Now observe that  $v_k^n = v_{\tilde{k}+1}^n + w_{\tilde{k}+1} \Delta x = v_{\tilde{k}+1}^{n+1} + \sigma_{\tilde{k}+1}^n \Delta t + w_{\tilde{k}+1} \Delta x$ ,

where  $w_{\tilde{k}+1} = \frac{v_k^n - v_{\tilde{k}+1}^n}{\Delta x}$ , and  $\sigma_{\tilde{k}+1}^n = \frac{v_{\tilde{k}+1}^n - v_{\tilde{k}+1}^{n+1}}{\Delta t}$ . By the

estimates in Section 2

$$|v_{\tilde{k}+1}^n - v_{\tilde{k}+1}^{n+1}| \leq C \Delta x ,$$

and therefore

$$\left( \frac{v_k^{n+\varepsilon}}{v_{\tilde{k}}^{n+1+\varepsilon}} \right) \leq 1 + C \frac{\Delta x}{\varepsilon} \leq C .$$

Consequently for  $\kappa_a^{(2)}$  we have

$$|\kappa_a^{(2)}| \leq C \|f\| (\Delta x)^\alpha \sum_{n_0}^N \sum_{\mathbf{Z}} \left| \frac{[Av]_k^n}{(\Delta x)^2} \right| \Delta x \Delta t$$

By Lemma 2.6

$$\begin{aligned} \sum_{n_0}^N \sum_{\mathbf{Z}} \left| \frac{[Av]_k^n}{(\Delta x)^2} \right| \Delta x \Delta t &\leq C \sum_{n_0}^N \left( 1 + \frac{1}{t_n} \right) \Delta t \\ &\leq C(1 + |\log \delta|) , \end{aligned}$$

so that

$$|\kappa_a^{(2)}| \leq C \|f\| (\Delta x)^\alpha |\log \delta|.$$

We estimate  $|\kappa_b^{(2)}|$  by performing a discrete integration by parts

$$\left| \frac{m}{m-1} \Delta x \sum_{n_0}^N \sum_{\bar{k}(n+1) \leq k \leq \bar{k}(n+1)} \left( \frac{w_{k+1}^n + w_k^n}{2} \right)^2 \left( \frac{\phi_{k+1}^{n+1} - \phi_k^{n+1}}{\Delta x} \right) \Delta x \Delta t \right| \leq$$

$$\leq C \Delta x \sum_{n_0}^N \sum_z \left| \frac{[Av]_k^n}{(\Delta x)^z} \right| |\phi_k^{n+1}| \Delta x \Delta t + C \Delta x \|\phi\| \leq$$

$$\leq C \|f\| (\Delta x)^\alpha |\log \delta|.$$

Combining these estimates we obtain

$$|H_4| \leq C \|f\| (\Delta x)^\alpha |\log \delta|.$$

By similar calculations involving the use of Lemma 2.6 we deduce

$$|H_5| + |H_7| \leq C \|f\| (\Delta x)^\alpha |\log \delta|.$$

In estimating  $H_6$  we first integrate by parts (discrete integration) and use the techniques above to obtain

$$|H_6| \leq C \|f\| (\Delta x)^\alpha |\log \delta|.$$

Analogous techniques give the desired estimate for  $H_8$ ,  $H_9$ ,  $H_{10}$ . The proof is complete.

Corollary 5.2. If  $-L \leq v_{0xx} \leq 0$ , for some positive constant  $L$ , then

$$|J(v^{h+\epsilon}, \phi^h, \delta, T)| \leq C \|f\| (\Delta x)^\alpha ; \quad \forall \delta > 0 .$$

### 5.3. Proof of Lemma 4.2

Since  $J(v^{h+\epsilon}, \phi, \delta, T) = J(v^{h+\epsilon}, \phi - \phi^h, \delta, T) + J(v^{h+\epsilon}, \phi^h, \delta, T)$  and the last term has been estimated in Lemma 5.1, we have only to estimate  $J(v^{h+\epsilon}, \phi - \phi^h, \delta, T)$ . To this end we will need the following preliminary fact

Lemma 5.3. There exists a constant  $C$  independent of  $n$ ,  $k$ ,  $\Delta x$ ,  $\phi$  such that

$$(\Delta x \cdot \Delta t)^{-1} \iint_{T_k^n} (\phi^h - \phi) dx dt \leq C \|f\| (\Delta x)^\alpha [1 + (\tilde{v}_k^n + \epsilon)^{-1}] ,$$

where

$$\tilde{v}_k^n = \min_{T_k^n} v^h$$

Remark An analogous statement holds for  $S_k^n$ .

Proof of Lemma 5.3:

$$\begin{aligned} \iint_{T_k^n} (\phi^h - \phi) dx dt &= \iint_{T_k^n} (\phi_k^n - \phi) dx dt + \\ &+ \iint_{T_k^n} [\psi_k^n(x - x_k) + \phi_k^n(t - t_n)] dx dt = I^{(1)} + I^{(2)}. \end{aligned}$$

For  $I^{(2)}$ , using (5.8)-(5.9) we have

$$|I^{(2)}| \leq C \|f\| (\Delta x)^\alpha [1 + (\tilde{v}_k^n + \varepsilon)^{-1}] \Delta x \Delta t.$$

Estimating  $I^{(1)}$ ,

$$\begin{aligned} I^{(1)} &= \iint_{T_k^n} \{[\phi_k^n - \phi(x, t_n)] + [\phi(x, t) - \phi(x, t_n)]\} dx dt \leq \\ &\leq \{\|\phi_x\|_{\infty, T_k^n} \Delta x + \|\phi_t\|_{\infty, T_k^n} \Delta t\} \Delta x \Delta t, \end{aligned}$$

and the desired estimate follows from (5.8)-(5.9).

Corollary 5.4: For all  $n, k$ , if  $\tilde{v}_k^n = \max_{T_k^n} v^h$

$$\frac{(\tilde{v}_k^n + \varepsilon)}{\Delta x \Delta t} \iint_{T_k^n} (\phi^h - \phi) dx dt \leq C \|f\| (\Delta x)^\alpha$$

$$\frac{\Delta x}{\Delta x \Delta t} \iint_{T_k^n} (\phi^h - \phi) dx dt \leq C \|f\| (\Delta x)^\alpha.$$

Remark: Corollary 5.4 holds if  $T_k^n$  is replaced by  $S_k^n$ .

We are now in the position to estimate  $J(v^{h+\varepsilon}, \phi - \phi^h, \delta, T)$ .

$$\begin{aligned}
 (5.10) \quad J(v^{h+\varepsilon}, \phi - \phi^h, \delta, T) &= \iint_{S_{T,\delta}} \{v_t^h (\phi - \phi^h) + m(v^{h+\varepsilon}) (v^h)_x (\phi - \phi^h)_x + \\
 &+ (m(v_x^h)^2 - \frac{m}{m-1} (v_x^h)^2) (\phi - \phi^h)\} dx dt = \\
 &= \sum_{n_0}^N \sum_{\mathbf{Z}} [\sigma_k^n - \frac{m}{m-1} (\frac{w_{k+1}^n + w_k^n}{2})^2] \iint_{S_k^n} (\phi - \phi^h) dx dt + \\
 &+ \sum_{n_0}^N \sum_{\mathbf{Z}} [\sigma_{k+1}^n - \frac{m}{m-1} (\frac{w_{k+2}^n + w_{k+1}^n}{2})^2] \iint_{T_k^n} (\phi - \phi^h) dx dt + \\
 &+ \frac{1}{4} \sum_{n_0}^N \sum_{\mathbf{Z}} (3w_{k+1}^n + w_{k+2}^n) (w_{k+1}^n - w_{k+2}^n) \iint_{T_k^n} (\phi - \phi^h) dx dt + \\
 &+ \frac{1}{4} \sum_{n_0}^N \sum_{\mathbf{Z}} (3w_{k+1}^{n+1} + w_k^{n+1}) (w_k^{n+1} - w_{k+1}^{n+1}) \iint_{S_k^n} (\phi - \phi^h) dx dt + \\
 &+ m \iint_{S_{T,\delta}} [(v^{h+\varepsilon}) (v^h)_x (\phi - \phi^h)_x + (v_x^h)^2 (\phi - \phi^h)] dx dt = \sum_{i=1}^5 p_i.
 \end{aligned}$$

Estimate of  $|P_1| + |P_2|$

$$\begin{aligned}
 P_1 &= \sum_{n_0}^N \sum_{\bar{K}(n+1) \leq k \leq K(n+1)} \left[ \alpha_k^n - \frac{m}{m-1} \left( \frac{w_{k+1}^n + w_k^n}{2} \right)^2 \right] \iint_{S_k^n} (\phi - \phi^h) dx dt + \\
 &+ \sum_{n_0}^N \sum_{[k < \bar{K}(n+1)] \cup [k > K(n+1)]} \left[ \sigma_k^n - \frac{m}{m-1} \left( \frac{w_{k+1}^n + w_k^n}{2} \right)^2 \right] \iint_{S_k^n} (\phi - \phi^h) dx dt = \\
 &= P_1^* + P_1^{**} .
 \end{aligned}$$

For  $P_1^{**}$  by the estimates in Section 2 and Corollary 5.4 we obtain easily

$$|P_1^{**}| \leq C \|f\| (\Delta x)^\alpha .$$

In estimating  $P_1^*$  we use the difference equation (1.11).

$$\begin{aligned}
 |P_1^*| &\leq C \sum_{n_0}^N \sum_Z \left| \frac{[Av]_k^n}{(\Delta x)^2} \right| \Delta x \Delta t \left\{ \frac{(v_k^n + \epsilon)}{\Delta x \Delta t} \left| \iint_{S_k^n} (\phi - \phi^h) dx dt \right| \right\} \\
 &\leq C \|f\| (\Delta x)^\alpha |\log \delta| .
 \end{aligned}$$

The estimate of  $|P_2|$  is analogous.

Estimate of  $|P_3| + |P_4|$ :

$$|P_3| + |P_4| \leq C \sum_{n_0}^N \sum_{\mathbf{Z}} \left| \frac{[Av]_k^n}{(\Delta x)^2} \Delta x \Delta t \left\{ \frac{\Delta x}{\Delta x \Delta t} \mid \iint_{S_k^n \cup T_k^n} (\phi - \phi^h) dx dt \right\} \right|.$$

By Corollary 5.4 and Lemma 2.6

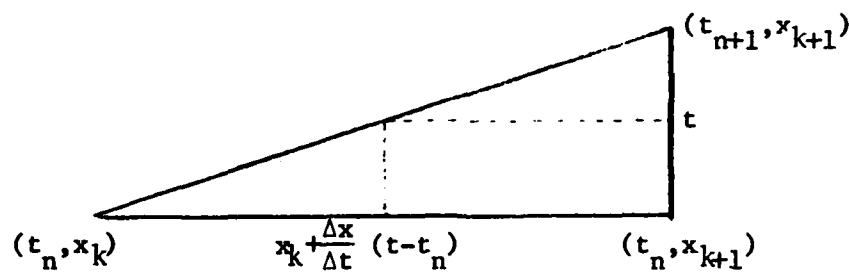
$$|P_3| + |P_4| \leq C \|f\| (\Delta x)^\alpha |\log \delta|.$$

Estimate of  $P_5$ :

$$\begin{aligned} P_5 &= m \sum_{n_0}^N \sum_{\mathbf{Z}} \left\{ \iint_{T_k^n} [(v^{h+\varepsilon}) v_x^h (\phi - \phi^h)_x + (v_x^h)^2 (\phi - \phi^h)] dx dt + \right. \\ &\quad \left. + \iint_{S_k^n} [(v^{h+\varepsilon}) v_x^h (\phi - \phi^h) + (v_x^h)^2 (\phi - \phi^h)] dx dt \right\} = Q^{(1)} + Q^{(2)}. \end{aligned}$$

We estimate  $Q^{(1)}$  by performing an integration by parts in  $x$  over each  $T_k^n$ .





We have

$$\begin{aligned}
 \iint_{T_k^n} (v^h + \epsilon) v_x^h (\phi - \phi^h) dx dt &= \int_{t_n}^{t_{n+1}} \int_{x_k + \frac{\Delta x}{\Delta t} (t - t_n)}^{x_{k+1}} (v^h + \epsilon) (v_x^h) (\phi - \phi^h) dx dt = \\
 &= \int_{t_n}^{t_{n+1}} (v^h + \epsilon) v_x^h (\phi - \phi^h) (t, x_{k+1}) dt - \\
 &\quad - \int_{t_n}^{t_{n+1}} (v^h + \epsilon) v_x^h (\phi - \phi^h) (t, x_k + \frac{\Delta x}{\Delta t} (t - t_n)) dt - \\
 &\quad - \iint_{T_k^n} (v_x^h)^2 (\phi - \phi^h) dx dt .
 \end{aligned}$$

Therefore

$$\begin{aligned}
 Q^{(1)} &= m \sum_{n=0}^N \sum_Z \left\{ \int_{t_n}^{t_{n+1}} (v^h + \epsilon) w_{k+1}^n (\phi - \phi^h) (t, x_{k+1}) dt - \right. \\
 &\quad \left. - \int_{t_n}^{t_{n+1}} (v^h + \epsilon) w_{k+1}^n (\phi - \phi^h) (t, x_k + \frac{\Delta x}{\Delta t} (t - t_n)) dt \right\} .
 \end{aligned}$$

Similar calculations for  $Q^{(2)}$  give

$$Q^{(2)} = m \sum_{n_0}^N \sum_{\mathbf{Z}} \int_{t_n}^{t_{n+1}} (v^{h+\varepsilon}) w_{k+1}^{n+1} (\phi - \phi^h)(t, x_k) dt + \\ + \int_{t_n}^{t_{n+1}} (v^{h+\varepsilon}) w_{k+1}^{n+1} (\phi - \phi^h)(t, x_k + \frac{\Delta x}{\Delta t} (t - t_n)) dt \quad .$$

For all  $k \in \mathbf{Z}$  and  $t \in [t_n, t_{n+1}]$

$$|(\phi - \phi^h)(t, x_k)| \leq C \|f\| (\Delta x)^\alpha (|\sigma_k^n| + 1) \Delta x$$

so that

$$\sum_{n_0}^N \sum_{\mathbf{Z}} \int_{t_n}^{t_{n+1}} (v^{h+\varepsilon}) w_k^n (\phi - \phi^h)(t, x_k) dt \leq \\ \leq C \|f\| (\Delta x)^\alpha \sum_{n_0}^N \sum_{\mathbf{Z}} (1 + |\sigma_k^n|) \Delta x \Delta t \leq \\ \leq C \|f\| (\Delta x)^\alpha |\log \delta| \quad .$$

Consequently

$$|Q^{(1)} + Q^{(2)}| \leq C \|f\| (\Delta x)^\alpha |\log \delta| +$$

$$+ m \sum_{n_0}^N \sum_{\mathbb{Z}} \int_{t_n}^{t_{n+1}} [w_{k+1}^{n+1} - w_{k+1}^n] (v^{h+\varepsilon}) (\phi - \phi^h) (t, x_k + \frac{\Delta x}{\Delta t} (t - t_n)) dt$$

Now from calculations analogous to the ones leading to (5.8) - (5.9), we have in  $T_k^n$  (and in  $S_k^n$ )

$$\begin{aligned} |(v^{h+\varepsilon})(\phi - \phi^h)| &\leq (v^{h+\varepsilon}) (\|\phi_x\|_{\infty, T_k^n} \Delta x + \|\phi_t\|_{\infty, T_k^n} \Delta t) \leq \\ &\leq C(v^{h+\varepsilon}) [1 + (v_{\tilde{k}}^{n+\varepsilon})^{-1} + (v_{\tilde{k}}^{n+\varepsilon})^{-1}] \|f\| (\Delta x)^\alpha \\ &\leq C \|f\| (\Delta x)^\alpha . \end{aligned}$$

Moreover

$$\begin{aligned} w_k^{n+1} - w_k^n &= \frac{1}{\beta} \left[ \frac{v_k^{n+1} - v_k^n}{\Delta t} + \frac{v_{k-1}^{n+1} - v_{k-1}^n}{\Delta t} \right] \Delta x \\ &= \beta^{-1} [\sigma_k^n + \sigma_{k-1}^n] . \end{aligned}$$

Hence

$$\begin{aligned} |Q^{(1)} + Q^{(2)}| &\leq C \|f\| (\Delta x)^\alpha |\log \delta| + C \|f\| (\Delta x)^\alpha \sum_{n_0}^N \sum_{\mathbb{Z}} |\sigma_k^n| \Delta x \Delta t \\ &\leq C \|f\| (\Delta x)^\alpha |\log \delta| . \end{aligned}$$

This completes the proof of Lemma 4.2.

## 6. Numerical Results

In this section we discuss briefly the results of our numerical experiments with the scheme (1.11). All computations were carried out on the CDC 6600 at Indiana University. In each case we specify only the value of  $\Delta x$  used. The values of  $\epsilon$  and  $\Delta t$  were always chosen to be the smallest and largest convenient values, respectively, consistent with the mesh conditions [A3] and [A4].

Not surprisingly, the condition [A4] on  $\Delta t/\Delta x^2$  is in fact necessary in practice. But the condition [A3] on  $\epsilon$  is probably overly restrictive. For example, for the specific problems discussed below, [A3] requires that  $\epsilon \geq 13.8\Delta x$ , which is not "small" when, say,  $\Delta x = .05$ . Indeed, for the second problem discussed below, we found that the accuracy increased noticeably as  $\epsilon$  decreased. A practical (but not theoretically justified) alternative to [A3] is the condition

$$(6.1) \quad \epsilon \geq \frac{m+1}{2(m-1)} \gamma_0 \Delta x .$$

Such a condition is sufficient for the bounds (2.1) and (2.2) for  $v^h$  and  $\frac{\partial v^h}{\partial x}$  to remain in effect. (The more stringent condition [A3] was required only for bound (2.15) for  $\frac{\partial^2 v^h}{\partial x^2}$ .) For the specific problems discussed below, condition (6.1) requires only that  $\epsilon \geq .58\Delta x$ .

For purposes of comparison, we used the Barenblatt-Pattle solution  $v$ , which for  $m = 2$  is defined by (see [16]),

$$(6.2) \quad v(x,t) = \begin{cases} \frac{1}{\zeta(t)} [1 - (\frac{x}{\zeta(t)})^2] & , \quad |x| \leq \zeta(t) \\ 0 & , \quad |x| > \zeta(t) \end{cases} ,$$

where

$$\zeta(t) = [12(t+1)]^{1/3} .$$

First, we applied the scheme (1.11) taking  $v_k^0 = v(x_k, 0)$ ,  $\zeta_\ell^0 = -\zeta(0)$ , and  $\zeta_r^0 = \zeta(0)$ . The computations were performed with three different sets of mesh parameters. Comparing the exact and computed solutions, at  $t = \frac{1}{2}$ , we found the following:

$\Delta x$	$ \zeta(\frac{1}{2}) - \zeta^h(\frac{1}{2}) $	$\ v(\cdot, \frac{1}{2}) - v^h(1, \frac{1}{2})\ _{\infty, \mathbb{R}}$
.1	.0202	.00666
.05	.0106	.00340
.025	.00551	.00173

Quite clearly, the observed errors in both  $\zeta^h$  and  $v^h$  are  $O(\Delta x)$ . This is significantly better than the rates predicted by Theorem 4.1, which are  $O(|\Delta x \log \Delta x|^{1/12})$  and  $O(|\Delta x \log \Delta x|^{1/6})$  for  $\zeta^h$  and  $v^h$  respectively. These discrepancies are explained by the fact that the solution (6.2) has derivatives of all orders which are uniformly bounded on its support, whereas the bounds (4.1) and (4.2) were derived under the minimal smoothness conditions which all solutions are known to satisfy. Another difference between the observed and theoretical results is that the observed rate of convergence for  $\zeta^h$  is the same as that for  $v^h$ , whereas Theorem 4.1 predicts roughly that  $|\zeta - \zeta^h| = O(\|v - v^h\|_\infty^{1/2})$ . However, the computation (4.15) shows that, as long as  $|\zeta(t) - \zeta^h(t)|$  is small,

$$|\zeta(t) - \zeta^h(t)| \leq \frac{C}{\dot{\zeta}_2(t)} \|v(\cdot, t) - v^h(\cdot, t)\|_{\infty, \mathbb{R}} + C \Delta x.$$

Thus when the interfaces are known to be moving with speeds bounded away from 0, the rate of convergence of  $\zeta^h$  will in fact coincide with that of  $v^h$ .

In the second example we took the same initial function  $v_0$  as before, but now with  $\zeta_\ell^0 = -3$  and  $\zeta_r^0 = 3$ . Thus  $v_0$  is neither concave nor continuously differentiable on  $[\zeta_\ell^0, \zeta_r^0]$ . The results were as follows:

$\Delta x$	$ \zeta(\frac{1}{2}) - \zeta^h(\frac{1}{2}) $	$\ v(\cdot, \frac{1}{2}) - v^h(\cdot, \frac{1}{2})\ _{\infty, R}$
.1	.1107	.06905
.05	.1073	.06697
.025	.1055	.06592

We are uncertain as to whether meaningful comparisons can be made between these data and (4.1) and (4.2). Nevertheless, it is clear that, at least qualitatively, Theorem 4.1 gives the correct result: in the absence of smoothness, the convergence may be quite slow.

The scheme (1.11) is thus seen to have two shortcomings. The first is that the parabolic stability condition [A4] makes it impractical to apply the scheme with small values of  $\Delta x$ . This difficulty can probably be overcome by employing instead a suitable implicit variant of (1.11). We intend to discuss such a scheme elsewhere. The other shortcoming of the present method is the unsatisfactory rate of convergence. While this phenomenon is partly due to the weakness of the exact solutions themselves, it may be possible to effect some improvement by a more sophisticated treatment near the interfaces.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) We study the convergence of a finite difference scheme for the Cauchy problem for the porous medium equation $u_t = (u^m)_{xx}$ , $m > 1$ . The scheme exhibits the following two features. The first is that it employs a discretization of the known interface condition for the propagation of the support of the solution. We thus generate approximate interfaces as well as an approximate solution.		

ABSTRACT (continued)

The second feature is that it contains a vanishing viscosity term. This term permits an estimate of the form  $\| (u^{m-1})_{xx} \|_{1,R} < c/t$ .

We prove that both the approximate solution and the approximate interfaces converge to the correct ones.

Finally error bounds for both solution and free boundaries are proved in terms of the mesh parameters.

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